Mathematics Library

OCT 14 1947

THE MATHEMATICS TEACHER

Volume XL · OCTOBER · 1947 · Number 6

CONTENTS

		Page
History of the Derivative and Integral of the Sine	Carl B. Boyer	267
Arithmetic for a Free Society	Ralph Beatley	276
The Measurement of Understandings and Judgments in Elementary School Mathematics (Arithmetic)	Ben A. Sueltz	279
Multi-Sensory Aids Based on Applications of Mathematics	Phillip S. Jones	285
The Art of Teaching Provision for Individual Differences in High School Mathematics Courses	William Lee	294
What Is the National Council of Teachers of Mathematics?		297
In Other Periodicals	Nathan Lazar	298
News Notes		299

OFFICIAL JOURNAL OF THE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
525 WEST 120TH ST., NEW YORK 27, N.Y.
Printed at Menasha, Wisconsin

Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorized March 1, 1930.

THE MATHEMATICS TEACHER

Official Journal of the National Council of Teachers of Mathematics

Devoted to the interests of mathematics in Elementary and Secondary Schools
Editor-in-Chief—WILLIAM DAVID REEVE, Teachers College, Columbia University.
Associate Editors—VERA SANFORD, State Normal School, Oneonta, N.Y.
W. S. SCHLAUCH, School of Commerce, New York University.

OFFICERS OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

President—CARL N. SHUSTER, State Teachers College, Trenton, N.J.
First Vice-President—H. W. CHARLESWORTH, East High School, Denver, Colo.
Second Vice-President—E. H. C. HILDEBRANDT, Northwestern University, Evanston, Ill.
Secretary-Treasurer—EDWIN W. SCHREIER, Western Illinois State Teachers College, Macomb, Ill.
Chairman of State Representatives—KENNETH E. BROWN, East Carolina Teachers College, Greenville, N.C.

ADDITIONAL MEMBERS ON THE BOARD OF DIRECTORS

One Year	LORENA CASSIDY, Wichita High School East, Wichita, Kan. ..1948
	HAROLD B. GARLAND, High School of Commerce, Boston, Mass. ..1948
	GEORGE E. HAWKINS, Lyons Township High School, La Grange, Ill. ..1948
Two Years	LEE BOYER, State Teachers College, Millersville, Pa. ..1949
	WALTER H. CARNAHAN, Purdue University, Lafayette, Ind. ..1949
	VERYL SCHULT, Wilson Teachers College, 11th and Harvard Sts. N.W., Washington, D.C. ..1949
Three Years	CHAS. H. BUTLER, Western State Teachers College, Kalamazoo, Mich. ..1950
	EMMA HESSE, Claremont Junior High School, Berkeley, Calif., 1950
	ONA KRAFT, Collinwood High School, Cleveland, Ohio ..1950

The National Council has for its object the advancement of mathematics teaching in elementary and secondary schools. All persons interested in mathematics and mathematics teaching are eligible to membership. The membership fee is $3 per year and entitles each member to receive the official journal of the National Council—THE MATHEMATICS TEACHER—which appears monthly except June, July, August and September.

Correspondence relating to editorial matter, subscriptions, advertisements, and other business matters should be addressed to the office of

THE MATHEMATICS TEACHER
525 West 120th St., New York 27, N.Y. (Editorial Office)

Subscription to THE MATHEMATICS TEACHER automatically makes a subscriber a member of the National Council.

SUBSCRIPTION PRICE $3.00 PER YEAR (*eight numbers*)

Foreign postage, 50 cents per year; Canadian postage, 25 cents per year. Single copies 40 cents. Remittance should be made by Post Office Money Order, Express Order, Bank Draft, or personal check and made payable to THE MATHEMATICS TEACHER.

PRICE LIST OF REPRINTS

	4pp. 1 to 4	8pp. 5 to 8	12pp. 9 to 12	16pp. 13 to 16	20pp. 17 to 20	24pp. 21 to 24	28pp. 25 to 28	32pp. 29 to 32
50 Copies	$2.50	$4.05	$6.35	$6.60	$ 8.55	$ 9.85	$11.40	$12.15
100 Copies	3.05	4.85	7.80	8.15	10.55	12.20	14.25	15.30
Additional copies per C.	1.05	1.60	2.65	3.15	3.60	4.70	5.75	6.30

For 500 copies deduct 5%; for 1,000 copies or more deduct 10%.
Covers: For first 50 covers, $3.00; additional 2½¢ each.
Plates printed on one side: For first 50 plates, $2.00; additional 1½¢ each. Printed on both sides, for first 50 plates, $3.00; additional 2½¢ each.
For more than 32 pages add cost per schedule to make total. Example: for 44 pages add cost for 32 pages and 12 pages.
NOTE: For any reprints requiring additional composition or changes in text or cover, an extra charge will be made.

THE MATHEMATICS TEACHER

Volume XL Number 6

Edited by William David Reeve

History of the Derivative and Integral of the Sine

By Carl B. Boyer
Brooklyn College, Brooklyn, N. Y.

The early development of the sine function was geometric rather than analytic. In ancient Greece the function concept and the ideas of differentiation and integration were not given formal expression, but the equivalents of these notions were implied by a number of problems in geometry. The determination of the length of a chord in a circle subtending a given angle x, for example, was the Greek equivalent of finding the value of the function $2 \sin x/2$. The method of exhaustion was the Greek geometric counterpart of finding an area by integration; and the kinematic determination by Archimedes of the tangent to his spiral is tantamount to the differentiation of a function. The calculus of the sine, however, could not at that time be given in terms of the now customary area and slope interpretations inasmuch as the sine curve was then unknown. Nevertheless, there are in the works of Archimedes several propositions analogous to the determination of the integral of the sine.

Preliminary to finding the surface and volume of a spherical segment, Archimedes proved[1] the following proposition:

Let AA' be the diameter of a circle and let QAQ' be any arc bisected at A. Let this arc be subdivided into $2n$ equal parts by the points $A, B, B', C, C', \cdots, P, P', Q, Q'$ (Fig. 1). Then

$$BB'+CC'+\cdots+PP'+QM):AM = A'B:BA.$$

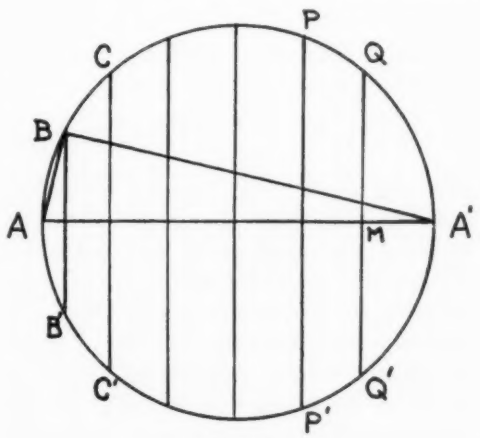


Fig. 1

[1] *On the Sphere and Cylinder* I, Proposition 22. See *The Works of Archimedes* (ed. by T. L. Heath, Cambridge, 1897), pp. 28–29; cf. pp. cxliv–cxlvi. For other aspects of the trigonometry of Archimedes see Johannes Tropfke, "Archimedes und die Trigonometrie," *Archiv für die Geschichte der Mathematik, der Wissenschaften, und der Technik*, X (1927–1928), 432–463.

This is the geometrical equivalent of the trigonometric equation

$$\sin\frac{\theta}{n}+\sin\frac{2\theta}{n}+\cdots+\sin\frac{(n-1)\theta}{n}+\frac{1}{2}\sin\frac{n\theta}{n}=\frac{1-\cos\theta}{2}\cot\frac{\theta}{2n}.$$

From this theorem of Archimedes it is a simple matter analytically to derive the modern expression $\int_0^\theta \sin x\,dx = 1-\cos\theta$ by multiplying both sides of the trigonometric equation above by θ/n and taking limits as n increases indefinitely. The left-hand side becomes

$$\lim_{n\to\infty}\sum(\sin x_i)\Delta x_i$$

$$\left(\text{where } x_i=\frac{i\theta}{n} \text{ for } n=1,\ 2,\ \cdots,\ n, \text{ and } \Delta x_i=\frac{\theta}{n} \text{ for } i=1,\ 2,\ \cdots,\ n-1 \text{ and } \Delta x_n=\frac{\theta}{2n}\right);$$

the right-hand side becomes

$$(1-\cos\theta)\lim_{n\to\infty}\left(\frac{\theta}{2n}\cot\frac{\theta}{2n}\right)=1-\cos\theta.$$

The equivalent of the special case

$$\int_0^\pi \sin x\,dx = 1-\cos\pi = 2$$

had been given by Archimedes in an earlier theorem. The language of Archimedes in his quadratures and cubatures follows the usual pattern of the ancient method of exhaustion, with both inscribed and circumscribed figures and with an argument by a reductio ad absurdum in lieu of the passage to the limit; but the fundamental notions are quite analogous to those formulated in the modern symbolism of the definite integral.

The infinitesimal methods of Archimedes were continued in antiquity by the Alexandrian mathematicians, including Pappus, and afterwards passed over into the Arabic civilization.[2] Meanwhile, the Greek trigonometry of chords, developed by Hipparchus and Ptolemy, was converted by the Hindus into the study of half-chords or sines, and this, too, was passed on to the Arabs. Through the Muslim civilization this knowledge was in turn transmitted to the Latin world, especially during the twelfth and thirteenth centuries. By the sixteenth century the works of Archimedes were well known in Europe, and one finds Cardan going over a problem on the circle similar to the ancient geometrical integration of the sine.[3] Early in the following century Kepler cited Cardan in this connection and used an arithmetical approximation to this integral in his *Astronomia nova* of 1609. In his calculations on the planet Mars he had occasion to add the sines of angles, for every degree, from 1° to 15°. This sum he found to be 2.08166, which when multiplied by an approximate value for $\pi/180$ gave him .03594. This, he noted, is a little more than the versed sine of 15°, which he took as .03407. Similarly he found the sum for 30° to be 7.92598, which, when multiplied as above "by the rule of proportional parts," becomes .13691, or a little more than vers 30° = .13397. Kepler found a similar agreement for 60° and 90° also.[4] Such calculations are similar to determinations of

$$\sum_{i=1}^{n}(\sin x_i)\Delta x_i,$$

where $x_i = i°$ and $\Delta x_i = \pi/180$, and hence are approximations to $\int_0^\theta \sin x\,dx = 1-\cos\theta$.

[2] See Heinrich Wieleitner, "Das Fortleben der Archimedischen Infinitesimalmethoden bis zum Beginn des 17. Jahrhundert," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Part B*, Studien, I (1931), 201–220.

[3] *De subtilitate*, XVI. See Hieronymus Cardanus, *Opera omnia* (10 vols., Lugduni, 1663), III, 593. This passage is given in Johann Kepler, *Opera omnia* (ed. by Ch. Frisch, 8 vols., Frankoforti A. M. & Erlangae, 1858–1870), III, p. 497, note 86. Citations of Kepler's work given in this paper are all based on this edition. Several volumes of a new edition of Kepler's works by Max Caspar have appeared recently.

[4] *Opera omnia*, III, 390–391. Cf. also pp. 105 and 335. For purposes of exposition the lan-

They thus represent an arithmetization of the geometrical proposition of Archimedes. However, whereas Archimedes had given, through the method of exhaustion, the equivalent of the *limit* of a sum, Kepler did not here link his result with limits or infinitesimals, and it is not clear just what significance he attached to his approximation.[5] Shortly afterwards, in attempting to calculate the magnitude of the attraction between the sun and the earth, he suggested a corresponding geometrical demonstration, in terms of infinitesimals, along the lines of one given by Pappus on volumes of spherical segments.[6]

Inspired largely by the works of Archimedes and Kepler, mathematicians of the second third of the seventeenth century rapidly expanded infinitesimal geometry until it culminated in the calculus. It was one of the leaders in this movement who, about 1635, took the next step in the integration of the sine. Roberval first reproduced the proposition of Archimedes on the sum of chords in a circle, putting it in the then current terminology of sines and

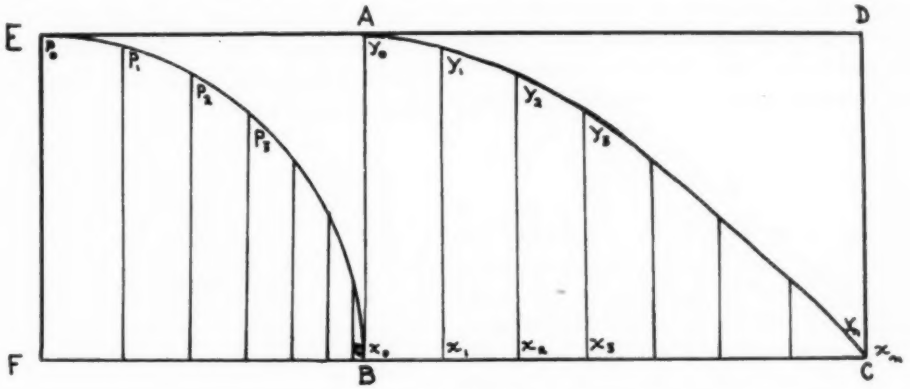


FIG. 2

infinitesimals: "If any arc of a circle is divided into an infinite number of equal parts and if it is projected orthogonally upon a diameter, the projection is to the arc as the sum of all the sines drawn from the points of division is to the product of the radius and the arc." The proof of this proposition, which is geometrically equivalent to $\int_{\theta_1}^{\theta_2} \sin x dx = \cos\theta_2 - \cos\theta_1$, is along the general lines of that given by Archimedes; but it differs in that the number of divisions is taken as infinite and an infinitesimal arc is substituted for an infinitesimal chord.[7] A corollary to this proposition gives the special case equivalent to $\int_0^{\pi/2} \sin x dx = 1$.

Roberval then gave a strikingly original proposition in which a portion of the sine curve appeared for the first time in his-

---

guage and notation of Kepler have been somewhat modified. Sines were lines in Kepler's day, and the ratio definitions of the trigonometric functions were not systematically used until the time of Euler. For a good general account of Kepler as a mathematician see Fritz Kubach, "Johannes Kepler als Mathematiker," *Veröffentlichungen der Badischen Sternwarte zu Heidelberg*, vol. XI, 1935. A briefer account is found in *D.* J. Struik, "Kepler as a mathematician," in *Johann Kepler, 1571–1630* (Baltimore, 1931), pp. 39–57

[5] For a full analysis with references see Gustav Eneström, "Ueber die angebliche Integration einer trigonometrischen Function bei Kepler," *Bibliotheca Mathematica* (3), XII (1912–1913), 229–241.

[6] *Opera omnia*, VI, 407. See Siegmund Günther, "Über eine merkwürdige Beziehung zwischen Pappus und Kepler," *Bibliotheca Mathematica*, new series, II (1888), 81–87; Gustav Eneström, "Sur un théorème de Kepler équivalent a l'intégration d'une fonction trigonométrique," *Bibliotheca Mathematica* (new series), III (1889), 65–66.

[7] See Evelyn Walker, *A Study of the Traité des Indivisibles of Roberval* (New York, 1932), pp. 79–81, 178–179; or G. P. de Roberval, "Divers ouvrages," *Mémoires de l'Académie Royale des Sciences depuis 1666 jusqu'à 1699*, VI (Paris, 1730), 1–478.

tory. Letting the line BC (Fig. 2) be equal to the quadrantal arc EB of a circle of radius $r = AB = FB$, he subdivided the line and the arc into n equal parts by points $x_0, x_1, x_2, x_3, \cdots, x_n$ and $P_0, P_1, P_2, P_3, \cdots, P_n$. Then at each of the points x_i he erected an ordinate x_iy_i equal in length to the perpendicular from P_i to the line FBC. Plotting the locus of the points y_i he obtained a curve which at the time was known as "Roberval's curve." It is, of course, half of one arch of the now familiar curve of sines.[8] Roberval then showed that the area under this curve is equal to r^2. This is easily deduced from the corollary above, for

$$AB : BC = \sum \overline{x_iy_i} : AB \cdot BC.$$

Clearing of fractions, the result follows from the fact that $\sum \overline{x_iy_i}$ is the area under the curve. Here one finds for the first time the interpretation of the definite integral of the sine in terms of the area under the sine curve; but it was more than a century before this became customary.

In 1659, just about a quarter of a century after Roberval's treatment of the sine, there appeared two small works devoted primarily to the integration of this function. One of these—the *Opusculum geometricum de linea sinuum et cycloide* of Honoré Fabri—illustrated the newer approach of Roberval; the other—the *Traité des sinus du quart de cercle* of Blaise Pascal—continued the older tradition of Archimedes. Referring only to a "certain anonymous celebrated geometer" as his inspiration, Fabri determined the area and center of gravity of the "figure of sines," as well as the volumes, surface areas, and centers of gravity of solids of revolution associated with the curve. Then he solved similar problems in connection with the cycloid, work that was overshadowed by Pascal's famous *Histoire de la roulette*.

In the *Traité des sinus* Pascal first restated the Archimedean result—"The sum of sines of any arc of a quarter of a circle is equal to that portion of the base included between the extreme sines, multiplied by the radius."[9] This theorem, equivalent to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin x_i \Delta x_i = \cos \theta_2 - \cos \theta_1,$$

where $x_0 = \theta_1$ and $x_n = \theta_2$, he easily proved in connection with a diagram (Fig. 3) which later became famous through the work of Leibniz. From the similarity of the right triangles EKE and DIA, and the resulting equality $DI \times EE = RR \times AB$, the

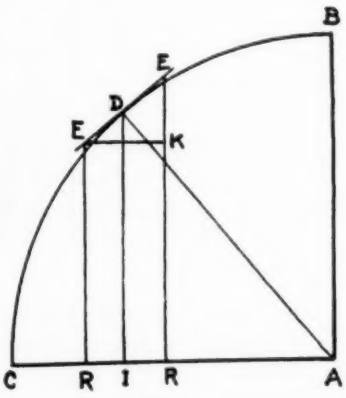


FIG. 3

theorem followed on making the intervals RR infinitely small and on substituting for the infinitesimal tangent line segments EE the infinitesimal arcs cut off by the ordinates ER. Pascal then went beyond the Archimedean result and showed similarly that the sum of the nth powers of the sines is equal to the radius times the sum of the $(n-1)$st powers of the ordinates. The theorem is equivalent to the equation

$$r^{n+1} \int_{\theta_1}^{\theta_2} \sin^n x dx = r \int_{r\cos\theta_2}^{r\cos\theta_1} y^{n-1} dx.$$

It is to be remarked that Fabri and Pascal considered only the definite inte-

[8] Walker, *op. cit.*, pp. 66–68, 180; Roberval, *op. cit.*, pp. 293–345. Roberval's diagram and notation have been somewhat modified for greater clarity.

[9] Blaise Pascal, *Lettres de A. Dettonville* (Paris, 1659), p. 1 f; or see his *Oeuvres* (ed. by Brunschvicg and Boutroux, 14 vols., Paris, 1908–1914), IX, 60–76.

gral. The indefinite integral or anti-derivative requires for its definition the formulation of the derivative, the basis for which was being laid at the very time these men were writing—and in connection with which Pascal played an inadvertent role. Their contemporary, Fermat, had developed an ingenious method of determining maxima and minima of a function $f(x)$. In this he first wrote $f(x+E)=f(x)$, and then, after simplifying by division and otherwise, he dropped out those terms which still contained E. This process is equivalent to finding

$$\lim_{E\to 0} \frac{f(x+E)-f(x)}{E}$$

and then equating this to zero. Fermat's method, which he applied also to the determination of tangents, is as closely analogous to differentiation as is the work of Archimedes to integration. Fermat and his contemporaries, however, seem to have been concerned largely with algebraic functions, and so the method was not applied directly to Roberval's curve of sines. The differential calculus of the sine function arose somewhat as had the integral long before—through the geometry of the circle.

Leibniz tells us that it was through a study of Pascal's work that he was led to his differential calculus.[10] Apparently it was in connection with the diagram for the *Traité des sinus* (Fig. 3 above) that Leibniz first realized that Fermat's method of differences was simply the determination of the ratio of two sides of the infinitesimal triangle EKE. Letting $RR=dx$ and $EK=dy$, the slope of the tangent at any point is thus given by dy/dx, or the ratio of two differentials. The differential or characteristic triangle played a prominent role in the calculus of Leibniz, and it was from this, through the obvious relationship $ds^2=dx^2+dy^2$, that the derivative of the sine arose. Although transcendental functions were not explicitly included in the early expositions published by Leibniz and Newton, it appears that the differential or fluxion of the sine was known from the earliest days of the calculus. A manuscript by Leibniz of 1676 includes the statement[11] that in a circle of radius r, if z is the arc of which x is the sine of the complement, and if in place of x one takes $x+\beta$ and in place of z one takes $z-dz$, then $dz=\dfrac{\beta r}{\sqrt{r^2-x^2}}$. This expression for dz is equivalent to $d \arccos \dfrac{x}{r}=\dfrac{-r\,dx}{\sqrt{r^2-x^2}}$.

Inasmuch as the characteristic triangle formed the basis of the differential method of Leibniz, it is probable that the result above, as well as the differential of the arcsine, was derived from the geometry of the circle. For any point P on a circle (Fig. 4) one deduces immediately, from the similarity of the triangles with sides dx, dy, dz and x, y, r, the results

$$dz=\frac{-r\,dx}{\sqrt{r^2-x^2}} \text{ and } dz=\frac{r\,dy}{\sqrt{r^2-y^2}}, \text{ where}$$

dz is $d \arccos x/r$ or $d \arcsin y/r$. In published works of Leibniz and the Bernoullis,[12] such diagrams and relationships appear quite incidentally, presumably because they were well known. There was at the time no clear distinction between dependent and independent variables, and so these equations appeared in various forms. The relationship $r^2dz^2=r^2dy^2+y^2dz^2$, for example, is easily converted into the inverse trigonometric form given above;

[10] See Dietrich Mahnke, "Neue Einblicke in die Entdeckungsgeschichte der höheren Analysis," *Abhandlungen der Preussische Akademie der Wissenschaften*, Physikalisch-Mathematische Klasse, I (1925), 1–64. Child, however, believes that Leibniz was led to his calculus by the work of Barrow rather than Pascal. See *The Early Mathematical Manuscripts of Leibniz* (transl. with notes by J. M. Child, Chicago and London, 1920).

[11] Leibniz (Child), *op. cit.*, p. 116 f.

[12] See *Acta Eruditorum*, III (1686), 297; VII (1691), 287, and Tab. VIII, fig. IV; X (1693) 179; XI (1694), 395–396, and Tab. XI, fig. 4. These are cited also in Enestrom, "Die erste Herleitung."

or, on writing it as $d\left(\frac{y}{r}\right)=\frac{\sqrt{r^2-y^2}}{r^2}\,dz$,

one may express it as $d \sin z = 1/r \cos z dz$ (where z is the arc) or as $d \sin \theta = \cos \theta d\theta$ (where θ is the central angle).

The calculus of the sine and arcsine were known also to Newton from the early days of his method of fluxions. He saw

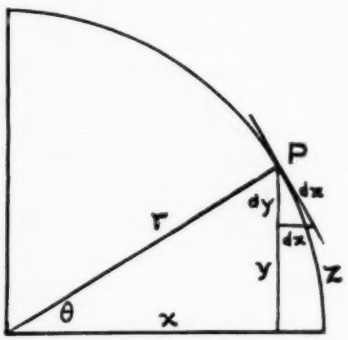


FIG. 4

likewise, through the characteristic triangle and the geometry of the circle, that $\dot{z}=\frac{r\dot{y}}{\sqrt{r^2-y^2}}$, where $\dot{z}$ is the fluxion of the arc corresponding to the fluxion $\dot{y}$ of the sine of the arc. (Fig. 4.) He then converted $\frac{1}{\sqrt{r^2-y^2}}$ into an infinite power series in y, using the binomial theorem.[13] On integrating this term by term, he obtained the series for arcsin y; and through his method for the reversion of series, the expansion for sin z was obtained. Using the relationship $r^2=x^2+y^2$, it was a simple matter for him to find the expansion for cos z by the binomial theorem. On applying the method of fluxions to the terms of these series, it was clear that the fluxion of the sine is given by the cosine, and that the fluxion of the cosine is the negative of the sine. These results were communicated to Oldenburg in 1676, the year of the Leibniz manuscript mentioned above, but Newton seems to have discovered them some years earlier.[14] James Gregory also arrived independently at these results at about the same time and in much the same manner,[15] and shortly afterwards Leibniz too was familiar with these series. It is interesting to note that whereas such series now are invariably derived through successive differentiation, they were obtained at that time through integration.

During the latter part of the seventeenth century infinite series played an increasingly important role in analysis, especially in connection with the trigonometric functions. Nevertheless, the trigonometry of the time remained a loosely organized study of geometrical propositions on lines in a circle. The ratio definitions had not yet entered; the trigonometric curves were scarcely known; and symbolisms, other than the conventional designation of lines by letters, were rarely used. It is not surprising, therefore, that the calculus of these functions was not at the time systematized. Inasmuch as trigonometric functions did not appear explicitly in equations, rules of differentiation and integration were not formally given. Where necessary, the fluxions or differentials of trigonometric lines were easily derived from the characteristic triangle or from the known infinite series representations. Moreover, trigonometric lines were regarded as functions of the arc rather than of the central angle, and hence the differentials of these were derived generally from the differentials of the inverse functions.

During the first half of the eighteenth century the role of the circular functions in the calculus was not greatly changed, but a new element, the idea of periodicity, became more persistent in trigonometry. The periodicity of the trigonometric functions generally has been ascribed to DeLagny who in 1705 indicated this for the tangent, but it is now known that the idea is at least thirty-five years older. The

[13] Newton, *Opera omnia* (ed. by Samuel Horsley, 5 vols., Londini, 1779–1785), I, 297 f.

[14] *Opera omnia*, I, 297 *f*, and IV, 451.

[15] James Gregory, *Tercentenary Memorial Volume* (London, 1939), pp. 6, 61, 155.

first clear-cut recognition of trigonometric periodicities seems to be that of Wallis in the *Mechanica* of 1670. Here two full cycles of the sine curve are clearly drawn; and the periodicity is evidenced by an accompanying statement that the sine curve increases throughout the first, fourth, fifth, eighth, ninth quadrants, *and so forth*.[16] Wallis seems to have been led to the idea of periodicity from the relation of the sine curve to the cycloid, where the repetition of cycles is more obvious to the eye from the manner in which it is generated. However, the periodicity of Wallis did not make a deep impression. Only much later, when the analytic goniometry of Euler showed that the multiple-angle formulas corroborated the idea, did the periodicity of the functions find general acceptance.

The brilliant young mathematician Roger Cotes was one of the few who in 1722 recognized the periodicity of the trigonometric functions, for he drew cycles of the tangent and secant curves; but his calculus of the functions remained essentially that of the previous century. He formalized the geometrical calculus of the cyclometric functions in the following lemma: "The least variation [or differential] of any circular arc is to the least variation of the sine of this arc as the radius is to the sine of the complement."[17] This is equivalent to the familiar result $d \sin \theta = \cos \theta d\theta$, where θ is the central angle corresponding to the circular arc whose sine is y. This is the clearest and most explicit statement up to that time on the differentiation of the sine, but the justification was along the traditional geometrical lines of Leibniz and the Bernoullis given above. Again in 1742, in Maclaurin's well known *Treatise of fluxions*, formal rules were given for powers, quotients, and logarithms, but the fluxions of the circular functions were still dependent upon the geometry of the characteristic triangle.[18]

The modern presentation of the trigonometric functions and their derivatives stems largely from the work of Euler. In his *Introductio in analysin infinitorum* of 1748 one finds for the first time a truly functional treatment of trigonometry. Influenced perhaps by the infinite series representations, Euler emphasized the numerical or analytic aspect of the subject, so that here the trigonometric functions are defined as ratios rather than lines. Clear graphs of the functions are drawn, showing their periodicities; and all the customary formulas of goniometry are given in terms of the now usual symbolic abbreviations.[19] However, Euler still defined the trigonometric functions in terms of the arc rather than the central angle.

During the half century following 1696 a number of textbooks on the differential calculus had appeared, but they did not include the trigonometric functions as such. The formal analytic treatment of the calculus of the sine begins with Euler's *Institutiones calculi differentialis* of 1755. Here the differential of the arcsine is first derived from that of the logarithm by means of the complex relationship

$$y = \text{arc sin } x = \frac{1}{i} \log (\sqrt{1-x^2}+ix).$$

This represents a continuation of the tradition which made the trigonometric functions subsidiary to others and which placed the inverse sine before the sine. Euler then added, however, that one can derive the differential of the arcsine more easily without logarithms as follows:[20]

[16] John Wallis, *Opera* (3 vols., Oxonii, 1693–1699), I, 504–505 and figure 201 opposite page 542.

[17] Cotes, *Aestimatio errorum*, p. 3. This is bound with his *Harmonia mensurarum* (Cantabrigiae, 1722), but is separately paginated. See *Harmonia mensurarum*, pp. 78–81, for the tangent and secant curves.

[18] Colin Maclaurin, *Traité des fluxions* (transl. by the R. P. Pezenas, 2 vols., Paris, 1749), I, 126 f.

[19] Leonhard Euler, *Introductio in analysin infinitorum* (2 vols., Lausannae, 1748). See especially I, 93 ff, and II, fig. 104.

[20] *Opera omnia* (23 vols., Lipsiae, 1911–1938), series 1, vol. X, pp. 132 ff. The deriva-

If $y = \arcsin x$, then $x = \sin y$ and $x + dx = \sin(y + dy)$. But $\sin(y + dy) = \sin y \cos dy + \cos y \sin dy$, and for an "evanescent" arc the sine may be taken as the arc and the cosine may be taken as unity. Hence $x + dx = \sin y + dy \cos y$, or $dx = \cos y dy$, or

$$dx = dy\sqrt{1-x^2}, \quad \text{or} \quad dy = \frac{dx}{\sqrt{1-x^2}}, \quad \text{or}$$

$$dy = \frac{dx}{\cos y}.$$

From the above it is obvious that the differential of $\sin y$ is $\cos y dy$; but before pointing this out, Euler preferred to derive "the inverse of the arcsine" directly by repeating the above argument for $y = \sin x$. This fact represents perhaps an early appreciation of the need for distinguishing between dependent and independent variables. He now wrote $y + dy = \sin(x + dx)$ or $dy = \sin(x + dx) - \sin x$, and for $\sin(x + dx)$ substituted $\sin x \cos dx + \cos x \sin dx$. Euler then replaced $\sin dx$ by dx and $\cos dx$ by 1, justifying this here in terms of the series expansions of the functions, and obtained $dy = \cos x dx$.

The treatises and papers of Euler exerted a decisive influence, and the trigonometric functions appeared regularly in calculus textbooks of the second half of the century. The Abbé Sauri in his *Cours complet de mathématiques* made special reference in an historical introduction to the fact that "Euler had added to analysis the new calculus of the sine and cosine"; and in the volume on the differential calculus he gave the Eulerian treatment of these functions.[21] When in the following year a French edition of the successful Italian calculus text of Maria Agnesi appeared, the editor added in an appendix the differentials of the sine and cosine, derived in the manner of Euler.[22] A half dozen years later a revised edition of the ever popular text of L'Hospital included an extensive note on the differential and integral of the sine and arcsine, derived geometrically first, and then analytically through Euler's identities.[23]

At the turn of the century the Eulerian calculus of the trigonometric functions was popularized by the ubiquitous textbooks of Lacroix; but even as these were appearing there was an insistent demand for a more rigorous treatment of analysis, free from Euler's infinitesimal zeroes. The wide search for a firm foundation led finally to the limit concept as the basis both in differentiation and in integration. Such an approach had been suggested by Newton, was more definitely indicated by D'Alembert and L'Huilier, and was made rigorous by Bolzano and Cauchy. It was the classic treatises of Cauchy in particular which led to the triumph of the limit concept, and in his *Résumé des leçons sur le calcul infinitésimal*[24] of 1823 one finds a thoroughly modern determination of the derivative of the trigonometric functions.

The inequality $1 > \frac{\sin\theta}{\theta} > \cos\theta$, for small values of θ, had been implied in the works of Archimedes, and Cauchy indicated that from this inequality it is clear that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ if the measure of the angle is in radins. Cauchy then formed, for $y = \sin x$, the difference quotient or "Cauchy fraction"

tion of the differential of the arcsine from the complex logarithmic relationship appeared also a year earlier in the *Traité du calcul intégral* of L. A. Bougainville (Paris, 1754), pp. 22–24. This work may well have resulted from the influence of Euler's many published papers.

[21] *Cours complet de mathématiques* (5 vols., Paris, 1774), I, xij; III, 36 ff. This work was reprinted at Paris in 1778.

[22] *Traités élémentaires de calcul différentiel et de calcul intégral* (Paris, 1775), p. 478 ff. The original Italian edition appeared at Milan in 1748, the year of Euler's *Introductio*.

[23] *Analyse des infiniment petits, pour l'intelligence des lignes courbes* (new ed. revised by Lefèvre-Gineau, Paris, 1781), pp. 30–31, 34–35.

[24] Augustin Cauchy, *Oeuvres complètes* (25 vols., Paris, 1882–1932), series 2. vol. IV, pp. 14, 22, 24.

$$\frac{\Delta y}{\Delta x}=\frac{\sin(x+i)-\sin x}{i}$$

$$=\frac{\sin \frac{1}{2}i}{\frac{1}{2}i}\cos(x+\tfrac{1}{2}i),$$

from which it was clear that the derivative (i.e., the limit as $i=\Delta x$ tends toward zero) is cos x. The derivative of $y=\arcsin x$ he derived in the now customary manner through the implicit function formula.

From the days of Newton and Leibniz to the time of Bolzano and Cauchy the integral had been defined as an antiderivative. In this interval the Archimedean summation concept had faded into the background, inasmuch as the inverses of fluxions and differentials are more easily determined than are the limits of sums arising in integration. The fundamental theorem of the calculus meanwhile was loosely justified by the simple geometrical observation that the rate of change of the area under a curve is proportional to the ordinate. It was largely the nineteenth century arithmetization of mathematics that led Cauchy to restore the summation concept of integration and to prove that the integral as so defined is the inverse of the derivative.[25] This proof, coupled with the work in differentiation, completes the modern equivalent of the ancient theorem of Archimedes on the sum of chords. In a sense, then, it marks the close of the historical development of the elementary calculus of the sine. During the first twenty centuries of this history trigonometry had remained essentially geometrical, and hence it played only an incidental role in infinitesimal methods. During the middle of the eighteenth century, however, it was quickly absorbed into the calculus after an arithmetization by which it became associated with the idea of functionality. This is an early illustration of a fact which has become increasingly evident ever since: the basis of trigonometry, of the calculus, and of all modern analysis is found in that central theme of mathematics—the function concept.

[25] Cauchy, *op. cit.*, p. 159 f.

Announcement of American Council on Education's 1948 Teacher Examination Program

Arrangements are now being made by the American Council on Education for the establishment of examining centers for the ninth annual administration of its National Teacher Examinations.

The examining centers are conducted in cooperation with school systems and teacher education institutions.

Superintendents and boards of education in many localities require teaching applicants to present National Teacher Examination records. The examination results are used as *one* of the factors in the selection of teachers.

The Teacher Examinations are also administered in connection with teacher education programs in colleges and universities, both at undergraduate and graduate levels. The examination profile is used for student guidance and self-study of strengths and weaknesses in areas measured by tests. The tests are frequently used as comprehensive examinations for undergraduates and as qualifying examinations for graduate students.

Used in combination with additional information such as that provided by records of experience, academic marks, ratings of various aspects of personality, etc., the Teacher Examination results can contribute materially to the attainment of education's fundamental goal, the provision of the best possible instruction for young people in the schools.

The American Council on Education welcomes the use of its examinations by any school system or college, *provided* assurance is given that the examination results will be *used wisely in combination* with other significant information concerning the prospective teacher.

Arrangements for the establishment of examining centers should be made by superintendents of schools and college officials before November 1, 1947. Correspondence regarding cooperation in the project may be addressed to David G. Ryans, Associate Director, National Committee on Teacher Examinations, American Council on Education, 15 Amsterdam Avenue, New York 23, New York.

Arithmetic for a Free Society

By RALPH BEATLEY
Harvard Graduate School of Education, Cambridge 38, Mass.

THE writer recently received the following inquiry from a teacher of sixth grade arithmetic. "I desire your opinion of the following method of problem solving, where the problems are those in which a certain relationship is given and another relationship is asked for. Below are samples."

A. How much will six tickets cost at 15 cents apiece?
B. If five tickets cost \$.60, how much will one cost?
C. How many tickets can you buy for \$1.50 if two of them cost \$.25?

"1-2-3" means the following three questions.

1. What is asked for?
2. What is given?
3. What is the thing to do?

Solutions of the above problems.

A. 1. 6 tickets = ? \$
   2. 1 ticket = \$.15
   3. 6 tickets = \$.15 × 6 = \$.90 (1 × 6)
B. 1. 1 ticket = ? \$
   2. 5 tickets = \$.60
   3. 1 ticket = \$.60 × $\frac{1}{5}$ = \$.12 (5 × $\frac{1}{5}$)
C. 1. \$1.50 = ? tickets
   2. \$.25 = 2 tickets
   3. \$1.50 = 2 tickets × 1.50/.25 = 12 tickets

The reply to this inquiry could have been confined to the relatively narrow limits envisaged by the inquirer. It could have told the teacher "what is the thing to do" with respect to this 1-2-3 method of getting the pupil to recognize in his turn "what is the thing to do." It requires no great imagination, however, to see that this simple inquiry is not merely a matter of sixth grade arithmetic, but involves an issue that concerns all teaching of mathematics at whatever level of instruction. To what extent ought anybody to tell anybody else "what is the thing to do"? And what ought the teacher to tell a pupil who wants to be told "what is the thing to do"? It is for this reason that the reply to this inquiry was framed with the broader issue in mind—and then submitted to THE MATHEMATICS TEACHER as possibly of interest to others than those to whom it was originally addressed. Not, however, to tell them what is the thing to do; but merely to raise the question in such a way that they will be incited to figure out for themselves the answer that seems to them most appropriate.

In answering this inquiry the writer might well have emphasized the value of understanding, meaning, and appreciation as aids to the pupil's learning of mathematics. He would detract nothing from these. Indeed, they are required to support the idea he now advances, which involves the very pith and core of mathematical instruction. This idea has also a broad social aspect. Usually when referring to the social aspect of mathematics we have in mind the application of mathematics in non-mathematical fields. Applications are important also. The writer hopes that his idea will be recognized as combining the purely mathematical and the broadly social, and will win wide application outside the mathematics classroom. This is what he wrote.

Dear———

I have been thinking about your question concerning the solving of problems in arithmetic. I enclose a brief list of references to articles that bear on the general question of problem solving in arithmetic and I commend them to you.

In addition to these, without turning now myself to these same articles for help in framing an answer to your question, but relying only on my own reflections

and on my own experience in trying to connect pupils with mathematics, I offer my own opinion.

I am dubious about approving any *system* for solving problems, however qualified by statements that the system is not compulsory but only suggestive. I incline to the belief that teachers who commend a system of the sort you quote do in effect urge it upon their pupils. I would not deny the value of a system to the extent that it serves to get a pupil *started* on the business of considering seriously the details of a problem that confronts him. But once he is started, there is no system that can help him over the crucial hurdle, which is "seeing the relations"; or, in the sample you sent me, that will help him to determine "what is the thing to do."

Every *genuine* problem has an element of novelty in it. Relatively few problems in school books are genuine in this sense. For the most part they follow certain types. Offhand I am loath to argue the relative merits of the system you quote and the system that encourages a pupil to identify the type under which a given problem falls. I can see merits in each; I recognize that problems can to some extent be typed; but what I chiefly see is the demerit in each. For if I were training pupils to solve genuine problems, I would aim to upset stereotypes by confronting pupils with genuine problems which differ sufficiently from the routine sorts to require new thinking each time. This thinking should turn around basic principles. The object is not to baffle and bewilder, but by properly graded stages to lead the pupil to rely upon these basic principles.

I am opposed to systems for this sort of learning, as I believe that the pupil is more likely to learn the system than to learn what the system aims to lead him to learn. Thus I may have no quarrel with the aim of the system, yet believe that the system will not help the pupil to attain the desired goal; for *the system will tend to supplant the goal.* I hope that the advocates of the system will be satisfied if I salvage the intended essence of it as follows.

> "The teacher ought to foster every habit of inquiry, every mode of examining the data, that can lead the pupil to come to grips with the salient mathematical relations in the problem before him. It does not matter if these devices themselves do not automatically suggest the nature of those relations, or indicate what the pupil ought to do with them in order to solve the problem. To my mind no device can really do these things, and it is well that it is so. But if I am wrong in this, and some device will actually give the pupil this aid, I would withhold it from him. I prefer that he have the point of view that it is up to him to discover the significant relations; that it is up to him to decide how to turn these to account. I want him to have the point of view that in a genuine problem no rigmarole will serve; that he must guard against acquiring improper confidence in rules-of-thumb and systematized procedures; that they will often mislead him; that there is no substitute for his own independent thinking."

As corollary of this last idea, there is no substitute for the confidence engendered by the satisfaction that the pupil gets when he has solved genuine problems by this entirely *self-reliant* method. The cue for the teacher is to supply a sufficient number of *genuine* problems that the pupil *has a good chance of solving* successfully *by his own thinking.* The teacher must judge how to maintain the balance between giving the pupil the necessary satisfaction that comes from successful accomplishment, and the whetting of interest and incentive that comes from the proposal of problems that stretch the pupil's powers.

Except for a reasonable number of these "stretchers," what good can come from giving pupils problems that they cannot eventually solve by their own thinking but are expected to "get" by means of a rigmarole provided for the purpose? We may seem by so doing to have given the pupil an ability he did not have before. I say instead that by so doing we rob him of something that he previously possessed —i.e., some bit of confidence in his own thinking—and, if we keep it up, we gradually reduce him to a person with *no* confidence in his ability to size up a situation that is really within his modest powers to grasp.

In this way we develop persons whose reaction to a brief paragraph of print that looks like a problem in arithmetic is avoidance or evasion. To themselves they say, "I probably can't get the answer to this one; I prefer not to be in the position of submitting an answer subsequently to be proved wrong; I will not risk loss of social esteem by getting involved in this; by some socially acceptable phrase I decline participation. If it is insisted that I try, I'll halfheartedly imitate a few procedures that I once saw used successfully in situations that looked something like this—though I never understood why they were successful; I may eventually shake out something that I will submit as an answer (though I myself see no reason why it is indeed the answer); my previous experience of failure in such situations prompts me to the belief that indeed it is not the answer; if I were to try really to see what the problem is about, and what sort of ideas are involved, I could not by my own efforts figure out anything acceptable." To themselves they admit that they are so unlikely to be confronted with a problem that will respond to their own unaided thinking, that they do not even try with the powers that they really possess—or once possessed. Openly, however, upon diffidently submitting their suspect solution, they say, "I once knew how to do problems like this, but I have forgotten now how it was that we did them."

From him who had little, even that little he had has been taken away. "Mein Gott, iss dees a seestem!" Assuredly yes; but what good is it? And does it not also do positive harm?

That a pupil should correctly appraise his limitations is part of understanding. That he should be trained to exercise his powers of thinking within these limits may be unusual in some quarters. But the unusual must become the usual if we are to train individuals to be free men rather than slaves. Any breakdown of personality in the direction of yielding to formula, trustingly applied but not understood, tends actually to slavery. This age cannot afford more of this. No age can afford any of it. All our efforts must be in the direction of building up self-reliant and confident use of all our powers, especially all of our power to think.

With proper modesty we may often mistrust the results of our thinking. Yet we must prefer the sometimes unfortunate consequence of our imperfect thinking to the general enfeeblement and eventual enslavement that will surely result if we abandon the effort to think for ourselves.

I do not here insist that mathematics is superior to other subjects for the purpose of training pupils to think. I insist rather that mathematics offers at least as many opportunities in the realm of thinking as do the other school subjects; opportunities that can be used to lead pupils to self-reliance, though often the result actually obtained is the opposite of this. I believe that we teachers of mathematics ought to make full use of these thousands of opportunities to encourage self-reliance in thinking, with this object clearly in view: that the most important characteristic of a free individual in a free society is willingness to think for himself, and that conditions in the world today require more than ever that the number of such individuals be greatly and steadily increased.

19th Yearbook

Be sure to get your copy of the new 19th Yearbook now!
The supply is limited.

The Measurement of Understandings and Judgments in Elementary School Mathematics (Arithmetic)

By Ben A. Sueltz
State Teachers College, Cortland, New York

I. Methods of Testing

Why do we test and measure in the elementary school? What useful purpose do we wish to serve? We may wish to secure a score or record indicating a mark of achievement or, on the other hand, we may be primarily interested in measurement in terms of its contribution to learning. It is this second function that I wish to emphasize. However, this is not intended to deny the worth of the former and its role in education. Dr. Thiele has discussed *Meaning* and *Understanding* in arithmetic. I want to describe methods of measurement of these important aspects of a modern school program.

Let me illustrate the close association of measurement and learning. Consider this block. (show cube) What is it? (cube) Take a good look at it. How many sides or faces has it? (six) What is the shape of each side? (square) (after placing the cube out of view) How many edges has the cube? (12) How many right angles are formed on the surface where the edges meet? (24) Note that this little exercise could be called measurement or it could be considered a method of instruction. Note also how the technique shifted from direct objective to mental-visual. We use similar techniques in instruction in shifting from objective materials to the abstract phases of mathematics. If we use different and progressive levels in instruction, then why should we refrain from doing so in testing and measurement?

In the discussion, I want to call attention to three types and levels of measurement that we have found useful in the measurement of understandings and judgments, namely measurement using:

1. Objective materials.
2. Pictorial and illustrated materials combined with verbal statements.
3. Normal school-room experiences.

Let us recall briefly the nature and content of the kind of mathematics we wish to measure. First of all it is functional. It is the kind that people should and can use in understanding the society about them. It includes all of the concepts, principles, relationships, information, facts, processes, and skills that an individual might fairly be expected to use if he is intelligent. It involves awareness or sensitivity, understanding and judgment, and should lead to a feeling of compulsion to react or to draw conclusions. I shall deliberately omit computations and problem solving as you have come to recognize these from the typical standard tests and school examinations.

II. Testing Using Objective Materials

1. Note these pieces of paper that were divided in half by a ten-year old girl. They illustrate progressively more complex understanding of the fraction a *half*. The pupil argued the correctness of those divided by crooked lines by reasoning that if something was added to one side of a middle line the same amount would have to be added to the other side. In response to the question "how many different ways can you divide the paper?", she responded, "lots and lots of ways." (A series of six illustrations were shown.)

You noted how closely the testing was associated with learning. Let me illustrate levels of understanding with these circles.

2. (Use two 4-inch circles and one 6-

inch circle made of compoboard) If these were cookies, how could you tell which is more, (a) the two 4-inch or (b) the one 6-inch cookie? They are made from the same thickness of batter. Pupils have suggested the following.

1. By estimating visually and by weighing
2. By superimposing, cutting and fitting paper circles
3. By counting the square inches of surface
4. By computing areas from the formula $A = r^2$
5. By mathematical comparison in terms of "the square of the corresponding dimensions"

Of course, even cookies like these have a much greater appeal to sensitivity, understanding, and compulsion to react than do mere paper circles in a textbook. Note that the two first methods suggested are experimental and are perfectly satisfactory at one grade level. But they are only relatively exact. The more mathematical procedures that involve computation require also a degree of what we might call mathematical understanding. Furthermore, the highest level of pupils can argue that these methods are useful for such extensions as "per cent of" and "per cent more than."

3. (use the parallelogram made of four splints with flexing pin joints) As I push this over (a lateral side) and make a parallelogram out of the rectangle, have I altered the perimeter? How can you tell? Have I altered the area? (show how accentuation of the principle leads even the more obtuse to a correct conclusion) Note again two levels of understanding? First we have the obvious visual impression which may have been in error and second we have the more advanced which led to a conclusion in terms of *base* and *altitude*.

For several years I have been experimenting with the use of objective materials in the measurement of understanding and judgment. But it is difficult to ascribe a score or number to this measurement and then defend it as reliable and valid. There are numerous variables such as types of materials, method of presentation, etc. We get different results when the materials are placed in the hands of the pupils than we do when a tested presents them to a whole class. I have found that conventionalized materials such as my paper cookies are better than real cookies for testing. Perhaps you recall Julian Huxley's story of the two Irishmen. Pat said to Mike, "Mike, what are you daydreaming about that should give you that smirk on your face?" To which Mike replied, "Women." Pat was a little puzzled and remarked that he thought it might be more satisfactory to have real women than just to daydream about them. But Mike knowingly responded, "I get a better class this way."

Even though we cannot obtain reliable objective scores from such testing we should continue its use because it is such a fine handmaiden of teaching and learning.

III. Testing Using Pictorial and Described Materials

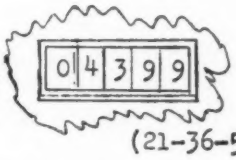
Next, I want to consider written tests designed to measure understanding and judgment. Note that the exercises that follow (pp. 281 and 282) are both pictorial and verbally described. These exercises are selected from more than 400 that have been used experimentally. Between 300 and 400 pupils have done each of the exercises. The numbers in parenthesis after each exercise show the percents of correct responses found in the several grades for which the tests were developed. Groups of pupils were interviewed in order to determine whether the exercises measured what was intended. The interviews proved to be much more valuable educationally than the use of statistical procedures alone.

Turn to Exercise No. 1 for Grades 4, 5, 6. Look at the results. Normally, in terms of curriculum expectations, we would expect to find much higher levels of learning. Some pupils actually set down the number 4399 and added one. Others

Selected Test Exercises**

Grades 4,5,6

1. The meter shows the number of gallons of water used. When another gallon is used, what number will the meter show?



(A) 4390 (B) 4400 (C) 4490 (D) 43991 (21-36-58)*

2. If you knew the total enrollment of your school and the total number present today, how could you find the number absent?

(A) add (B) subtract (C) multiply (D) divide (55-71-85)

3. What time is it by the clock at the right?

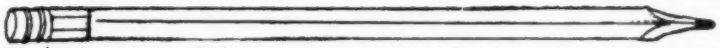
(A) 2:10½ (B) 3:52 (C) 2:53 (D) 10 to 3 (26-38-56)

4. Elsie plans to make sandwiches like the four shown at the right. How many sandwiches will she get from a loaf that has 20 slices of bread?

(A) 10 (B) 20 (C) 40 (D) 80

Gr.7,8,9 = (44-56-66) (21-29-20)

5. How many inches long is the pencil that is drawn below?



(A) 4 inches (B) 5 inches (C) 6 inches (D) 7 inches (49-46-52)

6. The pitcher holds 12 glassfuls. If glasses are filled only as far as the one at the right, about how many such partly filled glasses can be served from a pitcherful of lemonade?

(A) 10 (B) 12 (C) 16 (D) 24

Gr. 7,8,9= (82-85-91) (42-37-62)

7. Mrs. Barr used one-third of her bottle of medicine in 6 days. At this rate, how long should the remainder of the medicine last?

(A) 2 days (B) 3 days (C) 9 days (D) 12 days (45-59-69)

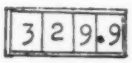
8. There are 18 girls in each of four classes of pupils. Which class below has the largest fraction of girls?

(A) 36 pupils (B) 33 pupils (C) 29 pupils (D) 40 pupils (16-29-32)

9. Which of the following divisions is correct?

(A) $4\overline{)121}$ $\quad 31$ (B) $4\overline{)121}$ $\quad 3\frac{1}{4}$ (C) $4\overline{)121}$ $\quad 05\frac{1}{4}$ (D) $4\overline{)121}$ $\quad 30\frac{1}{4}$ (38-47-71)

10. The mileage on a speedometer is as shown at the right what number will it show when another tenth of a mile is added?



(A) 330.0 (B) 329.0 (C) 329.91 (D) 330.9 (14-28-46)

** From experimental tests developed by Ben A. Sueltz

* Per cent correct in gr. 4,5,6 respectively

Selected Test Exercises

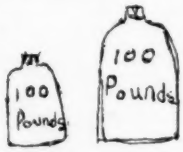
Grades 7,8,9

1. When eleven days in a month are cloudy, for which of the following months would this be the largest fraction?

   (A) January (B) February (C) March (D) April (47-50-76)*

2. Mary was dividing by 28, the first figure in the quotient was 5, and the remainder was 26. What would be the next figure in the quotient after she "Brought down" a figure?

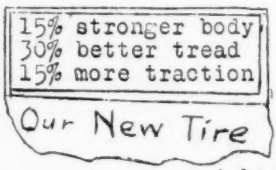
   (A) 1 or 2 (B) 4 or 5 (C) 8 or 9 (D) You can't tell (11-15-40)

3. 

coal feed

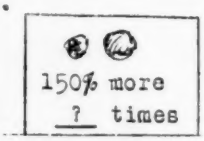
   How does the number of 100 pound coal-bags that can be filled from a ton of coal compare with the number of 100 pound feed-bags that can be filled from a ton of chicken feed?

   (A) the same (B) a few less (C) a few more
   (D) about half as many (28-24-50)

4. 

   A new tire is advertised using per cents as shown at the left. By what percent is the new tire better than the old tire?

   (A) 60% (B) 20% (C) 22½%
   (D) You can't tell exactly (16-8-10)

5. 

   A home economics worker noted that large oranges contain 150% more juice than small oranges. How many times as much juice do the large oranges contain?

   (A) 2 times (B) 1½ times (C) 2½ times
   (D) 1 and 1/3 times (11-6-15)

6. Why do scientists prefer to use decimal fractions rather than common fractions in their work?

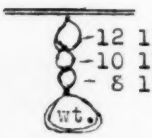
   (A) decimals are easier to use in figuring
   (B) most rulers have decimals (C) decimals are older (51-31-60)
   (D) decimal tenths are more exact than common fraction tenths

7.

	seeds	plants
Corn A	5	4
Corn B	10	8
Corn C	50	40

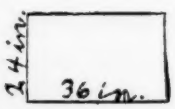
   The chart at the left shows how many plants sprouted from three different samples of seed corn. Which is most likely to produce 80 plants from each 100 seeds in a bushel of corn?

   (A) corn A (B) corn B (C) corn C
   (D) all three samples are equally likely (39-24-25)

8. 

   A weight is attached to the bottom end of three links of chain. The links are each strong enough to hold the number of pounds as marked. How heavy a weight will the chain hold without breaking?

   (A) 8 lb. (B) 10 lb. (C) 12 lb. (D) 30 lb.
   (87-79-64)

9. 

   The box top is to be covered with a single piece of metal with no seams. Which of the pieces below can be used with the least waste? (43-47-70)

   (A) 26 in. by 34 in. (B) 22 in. by 40 in.
   (C) 26 in. by 36 in. (D) 24 in. by 38 in.

* Per cent correct in grades 7,8,9 respectively

grasped the significance of the exercise when the statement "when another gallon is added" was inserted. Interview showed that many pupils had not extended the "number sequence" idea into numbers as large as thousands. I have found that the use of a "tally counter" or old car speedometer is valuable for developing the sequential relationship of numbers.

Exercise No. 2 is a typical relationship of a mathematical situation to a process. As a whole, results for simple single-process situations were satisfactory.

A surprising number of pupils chose answer (A) $2:10\frac{1}{2}$ for exercise No. 3. You may wish to argue that this is an unfair alternate choice. Visitation of schools and interviews with pupils showed that many classrooms do not have clocks visible to pupils and that there is a tendency to neglect extensions of "telling time" and work with *time* beyond the simple stages usually found in the first three grades.

Note exercise No. 4. What is the correct answer? How did you get it? Some pupils went through several stages of written computation while others obtained correct answers merely by a little resourceful reasoning. This and other exercises have led me to believe that formal computations frequently are a stumbling block to the development of real understanding and judgment.

It is a little sad to report that college freshman are only a little better than grade six in judging the length of the pencil in exercise No. 5.

Exercises Nos. 6, 7, and 8 include ideas involved in understanding fractions. In Ex. No. 6, formal computational procedures again were a hindrance to meaningful understanding. Exercise No. 8 was difficult largely because the pupils did not sense that this was merely an extension of a principle that they had learned with abstract numbers.

Exercise No. 10 is included in this list to show the similarity of progression of the per cents correct with exercise No. 1.

Let us consider briefly the exercises for grades 7, 8, and 9 on the following page.

Exercise No. 1 is similar to No. 8 on the preceding page.

Exercise No. 2 deals with the understanding of a mathematical relationship in the division process. Since many teachers have not developed this understanding, it is not surprising that pupils are similarly weak. This is the kind of thing we do not find in textbooks and which many of us believe should become a sensitivity to mathematics developed through meaningful teaching.

In interview, pupils argue about exercises Nos. 3 and 4. Here also their desires to compute caused trouble. A number of them said that you have to add in No. 4 because there are three numbers.

Exercise No. 5 uses an extension of the ratio idea of fractions. The results suggest that we need to pay attention to exactness of language as well as to exactness of computation.

In exercise No. 6, answer (D) proved very popular. However, when pupils were counseled, those of average and better ability soon sensed the error of this choice.

In exercise No. 7, I was trying to test the idea of "relative reliability in terms of the size of the sample." Interview showed that the pupils did not grasp this. It is not a good exercise for this age-group of pupils.

The popular wrong answers to exercise No. 8 were (B) and (D). The elements of understanding and judgment were glossed over by pupils who wanted to compute.

Exercise No. 9 is a "honey." Can you do it?

Thus far I have illustrated measurement first by use of objective materials and then by using printed materials that feature illustrations as well as verbal statements. In this latter procedure I pointed out how it was interviews that frequently gave the real clues to understanding or the lack of it. From these test exercises it is possible to obtain a score for record purposes, but the suggestions for teaching, and especially for the non-meaningful

teachers, may best be gleaned from discussion with pupils. Also, I should like to point out that interviews showed that, on tests like these, there is almost no pure guessing.

IV. Informal Testing

Finally, I wish to discuss briefly measurement based upon normal school-room procedures. The good teacher, the one who really understands mathematics and its functional uses as well as the pupils with whom she deals is constantly observing and counseling with her pupils. She knows when her pupils understand. She can even rate them, as for example, A, B, C, D in understanding. It is in her room that pupils frequently say, "I understand now." Furthermore, this teacher can, at any instant, ask the question that is at the same time a test question and also a teaching question. This type of teaching is beyond the mechanical, it is far above the "workbook level" of learning, and considerably superior to routine textbook procedures. Meaningful and functional mathematics requires the use of real clocks, rulers, potatoes, bottles, etc.

Measurement and evaluation as a part of the daily work of the teacher does not require a special technique. Rather it requires a point of view and a willingness plus a thorough understanding of the mathematics of the elementary school and its role in functional society. This casual and yet deliberate part of the teacher's work is interwoven with teaching and learning. It is difficult to describe but the capable teacher or superviser recognizes it at a glance. The teacher works with her pupils, she knows that each one must learn as an individual and hence she watches the work of each and directs it so that genuine meaning and understanding become realities.

To summarize, measurement of *understanding* and *judgment* in arithmetic and mathematics are possible. Three methods have been described. In each of these, emphasis was placed upon the role of the teacher and close tie between measurement and learning. A high quality of teacher education is necessary. This education seems to fit the older pattern of professionalized subject matter rather than courses in educational theory or advanced courses in college mathematics.

Multi-Sensory Aids Based on Applications of Mathematics*

By PHILLIP S. JONES
University of Michigan, Ann Arbor, Mich.

A UNIFYING theme for this discussion of mathematical models may be found in the three words, *concreteness*, *reality*, and *mathematics*. Any multi-sensory aid of course gives a psychologically desirable concrete embodiment to more abstract concepts; aids based on applications give an added reality that increases interest and attention, but none of these results is of real value unless they combine to produce a better understanding of substantial, worthwhile mathematical principles. Having stated these principles, we will, in the interest of time saving and conciseness, leave it to you to evaluate the devices and determine how or where they may be of use, and we will here proceed to the discussion of specific devices.

Since one should at the outset seek to

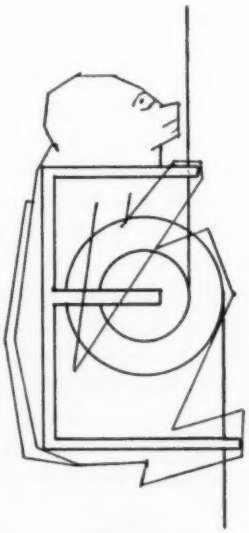


FIG. 1.

gain the attention of an audience, perhaps we will do well to start with this (demonstrating a bright red toy monkey which appears to slide up and down a string as the lower end of a cord is pulled. See Fig. 1 for a schematic diagram). A first question would naturally be "how does it

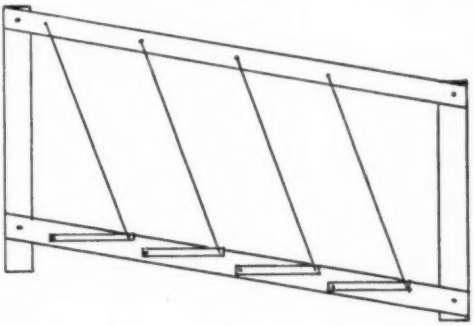


FIG. 2.

work?", and this brings out the physical as well as mathematical concepts involved in a set of differential pulleys and the mathematics of turning moments. A second question would be "how far does he move and by how much does the string appear to lengthen?" This can be investigated in terms of the radii and circumferences of the two pulleys.

Remaining for a few moments on a still quite elementary level, let us note the parallelograms and triangles of this adjustable gang-plank[1] (Fig. 2), the steps of which will always remain horizontal as the outer end of the "gang-plank" is raised or lowered to meet deck levels varying with the ship or the height of the tide.

Parallelograms are put to another use in constructing this spacing divider (Fig. 3) which may be used by an aerial navigator to divide a line segment into any desired number of equal parts from two to ten, and thus to locate on his chart

* Read March 1, 1947 at the 25th Annual Meeting of The National Council of Teachers of Mathematics at Atlantic City, N. J.

[1] Brown, H. T., *507 Mechanical Movements*, Brown Coombs and Co., 1871, p. 95.

"landmarks" to be sought at equal intervals of time (or distance) when navigating by piloting.[2] The Germans of course give this device a wonderfully compounded name, Teilungsmasstabe, add to it a ver-

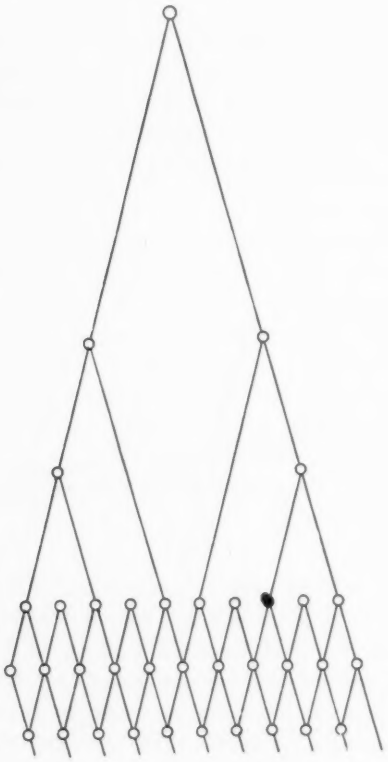


FIG. 3. Spacing Divider.

nier and make of it a precision instrument.[3] In the opposite direction as to refinement, one finds a simplification of this device termed a "hubagraph" (Fig. 4) and used to determine the spacing of lettering and rulings on the back of a newly rebound book.[4]

Obviously this device is also related to the familiar pantagraph, which will be mentioned again later, but which is shown in this picture (from *Life* magazine) in an industrial application to the simultaneous cutting of eight congruent shapes from a single piece of metal by a battery of oxy-acetylene torches.

This advertisement (of Columbia Records) shows in an unusual fashion an example of a useful geometric concept. namely, the ruled surface which has here been traced, supposedly, at least, by the

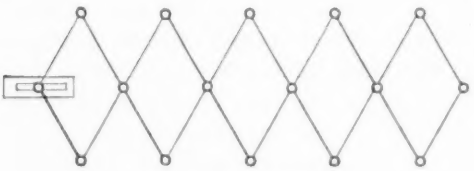


FIG. 4. Hubagraph.

illuminated baton of the orchestra conductor shown in the picture.

One of the most common and simple of these ruled surfaces is the hyperboloid of revolution. The justification for discussing this surface again at a meeting where it has been mentioned and displayed often before may perhaps lie in the use of elastic thread which makes it possible to show with a simply constructed and operated model the variation from cylinder to hyperboloid to cone. By fixing in place the adjustable end-circle, this hyperboloid may be mounted so as to roll on the congruent hyperboloid below it. The two hyperboloids are in contact, of course, along the entire length of their rulings and demonstrate a friction device for transforming rotation about one axis to a corresponding rotation about an axis oblique to the original.[5]

In a recent speech which was influential in transforming into activity the British

[2] Lyons, Thoburn C., *Practical Air Navigation*, Civil Aeronautics Bulletin No. 24, 1940.

While stopping off at the Franklin Institute on the return trip from Atlantic City, the author saw displayed there a stainless steel spacing divider made by Theo. Alteneder and Sons of Philadelphia; Keuffel and Esser also make them.

[3] Dyck, Walther, *Katalog Mathematischer und Mathematisch-physikalischer Modelle, Apparate und Instrument*, Munchen 1892, p. 225.

[4] Perry and Baab, *The Binding of Books*, Manual Arts Press, Peoria, 1940, p. 138–139.

[5] Hilbert, D. and Cohn-Vossen, S., *Anschauliche Geometrie*, Dover Publications, 1944, p. 253.

Association's interest in multi-sensory aids, A. P. Rollet further cited the hyperboloid as the proper shape for one type of ploughshare.[6] A mathematician may well regret the passage of the old type battleship with its basket masts which also recalled the same surface.

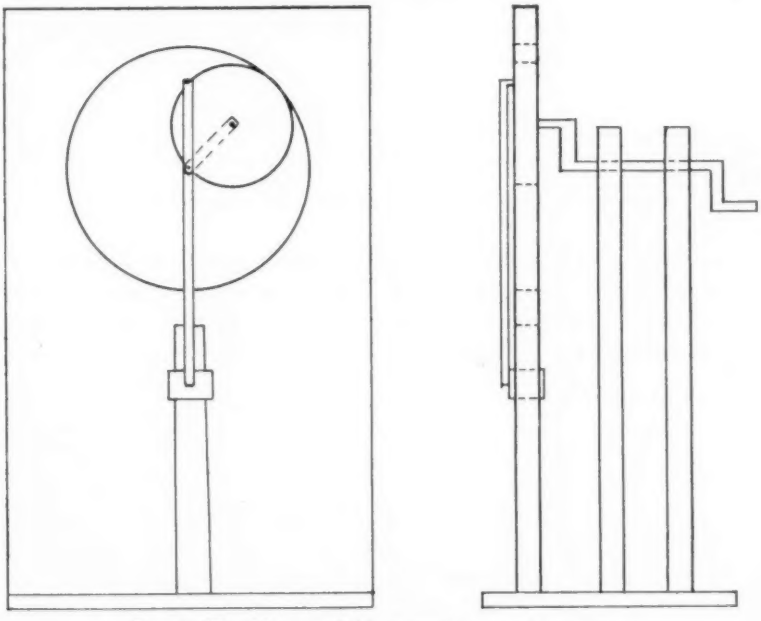


FIG. 5. The Hypocycloid as used in an old pump.

In the *Eighteenth Yearbook of the National Council of Teachers of Mathematics*, Dr. Baravalle made use of the hyperboloid of revolution to derive a construction for the hyperbola as the envelope of a family of straight lines.[7] Viewing this string model from another direction reveals a circle as an envelope of straight lines.

This talk of conics and the transmission of rotary motion recalls to our minds elliptic gears which have been discussed often enough before,[8] but of which this model may be worth pausing to consider. Made from two automobile brake pedal pads, it not only is easily carried to class rooms, but at the same time reemphasizes the frequency with which this curve is seen and put to use in the world around us.

Talk of ellipses and gears suggests the hypocycloid, the locus of a point on the circumference of a circle rolling on the inside of a second circle. The special hypocycloid in which the diameter of the inner circle is half that of the outer was used in an early pump now in the Edison Institute Museum at Dearborn, Michigan. In this special case, the hypocycloid is a straight line. (In his *Mathematical Snapshots*, Steinhaus included a set of cards which when riffled showed this case nicely.) In the pump mentioned of which this (Fig. 5) is a model, this feature of the hypocycloid was used to convert the rotary motion of a shaft into linear motion of a piston. Note that this is *not* the crank and connecting rod mechanism now commonly used. (The latter also has considerable of mathematical interest.) The fact that if a pencil is attached to any point on the inner circle it will draw an ellipse on a cardboard held in contact with it explains why this model may also be used to ex-

[6] Rollet, A. P., *Mathematical Models and Constructions*, Mathematical Gazette, Dec., 1945, 29: 185.

[7] National Council of Teachers of Mathematics, *18th Year-book:* 1945, H. V. Baravalle, *Geometric Drawing*, pp. 64–81, especially p. 79.

[8] *Ibid.*, P. S. Jones, *Mathematical Apparatus*, pp. 212–225, especially pp. 212 and 224.

plain the mechanism of some types of ellipsographs manufactured commercially for draftsmen and engineers.

An off-shoot of another member of the cycloid family may be drawn by fastening a pencil to any point within a circle which rolls on a straight line. This pencil will trace a trochoid which, upside down from its position as shown in Fig. 6, is used by naval architects as the best approximation to the shape of ocean waves. This curve is therefore fundamental in the study of such properties as the stability and angle of roll of ships.[9]

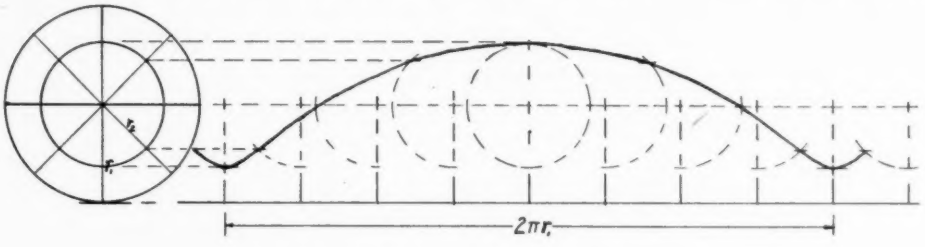


FIG. 6. The Trochoid, a draftsman's construction.

Naval architects make use of another old and honorable curve, the cissoid, in the design of planing hulls such as the PT boats of recent fame. This curve may be defined either as the locus of all points P such that $OP_i = C_iL_i$ as in Fig. 7, or (see Fig. 8) as the locus of all intersections Q_i of lines radiating from O with the horizontal lines R_iQ_i. These R_iQ_i are determined as symmetric to the lines S_iT_i with respect to the axis of symmetry of the semicircle. Devices may be constructed to duplicate either of these definitions mechanically and so to draw the curve.[10] J. C. Sturm, as shown in slide 1, did this for the second definition in his book *Mathesis Enucleata* published in Nuremburg in 1689

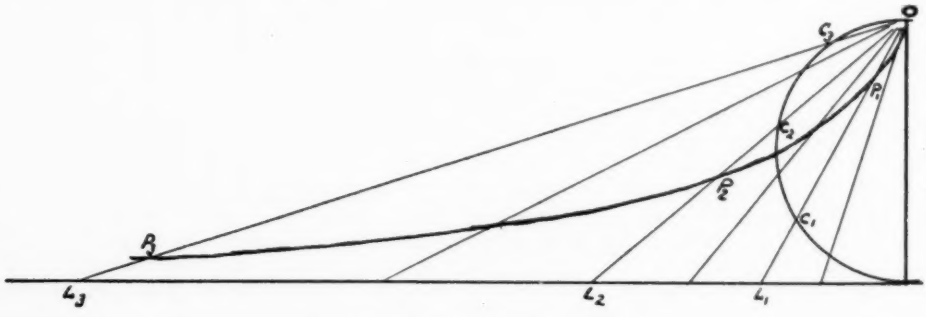


FIG. 7. The Cissoid of Diocles.

from the copy of which in the University of Michigan Library the slide was made. Of course he was not the originator of this curve; the credit for this goes to Diocles (c. 100 B.C.) who used it to solve the problem of duplicating the cube.[11] The naval

[9] See: Atwood, E. L. and Pengelly, H. S., *Theoretical Naval Architecture*, Longmans Green and Co., 1931, pp. 228 ff., 290, or other editions of this or other texts in this field.

[10] The drawings displayed at this time were made to show the step-by-step construction of the curve by superimposing upon one another a sequence of partial drawings made with Blaisdell's colored "Cellophane" pencils on transparent "Tufilm" made by M. Grumbacker, New York. This technique was originally discussed by Professor L. S. Johnston of the University of Detroit at the 1946 Fall Meeting of the Michigan Section of the Mathematical Association of America.

[11] For discussion of the cissoid (and trochoid also in the last two references) see:

architects used this curve as a close and satisfactory approximation to the experimentally determined optimum shape for a longitudinal section of the hull. To quote Lindsay Lord:[12] "There have been serious attempts to generate hull lines by a mathematical formula, but for practical purposes it is evident that the merit of such procedure lies very largely in the resulting similitude for duplication in other sizes according to whatever shape was first worked out, rather than in achieving any possible virtue inherent in a mathematical formula—by judicious application of logical equations to the development of the more important running lines of a known hull good similitude is retained throughout a range of sizes and assures predictable performance." Developable surfaces are used "when the bottom or topsides are to be sheathed with single sheets of plywood or other stiff material which cannot be bent into compound curves."

As you see, ruled surfaces which are also developable come up for application again here. The development of similar uses of conic sections and the projective constructions for them in the lofting rooms of large aircraft factories during the war should be familiar to both secondary and collegiate teachers.[13]

It was interest in this latter development which partially motivated the speaker's interest in the mathematical theory of perspective. Although the *18th Yearbook* contained two interesting articles[14] which discussed perspective and one reference[15] on this topic none of them made clear that there is an exact mathematical theory of perspective with a long and interesting history, modern applications and substantial mathematical connections of both an elementary and a more advanced nature. All of these facts can be only briefly developed here with the hope of doing them real justice at a later date.

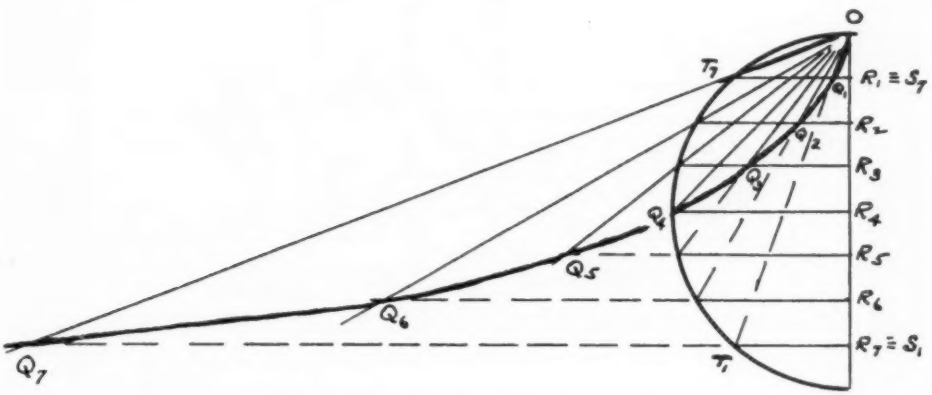


FIG. 8. The Cissoid, second definition.

The elements of any problem in perspective are an object, an eye which observes the object, and a surface cutting the cone of rays formed by joining the eye to all

---

Heath, T. L., *A Manual of Greek Mathematics*, Oxford, 1931, p. 167.

Lord, Lindsay, *Naval Architecture of Planing Hulls*, Cornell Maritime Press, 1946, p. 77.

Merriman, G. M., *To Discover Mathematics*, John Wiley and Sons, 1942, p. 127.

Yates, R. C., *Curves* 1946, pp. 225 ff., pp. 27 ff. U.S. Military Academy. Now published by Edwards Bros., Ann Arbor, Mich.

Yates, R. C., *Tools*, Louisiana State University, 1941, pp. 158, 184.

[12] Lord, Lindsay, *Ibid.*, p. 77, 82.

[13] See for example Liming, R. A., *Practical Analytic Geometry with Applications to Aircraft*, Macmillan, 1944.

[14] *Loc. cit.*, Hartley, M. C., *A Laboratory Approach to Solid Geometry*, pp. 42–63, and Baravalle, H. V., *Geometric Drawing*, pp. 64–81.

[15] Norling, Ernest, *Perspective Made Easy*, Macmillan, 1939.

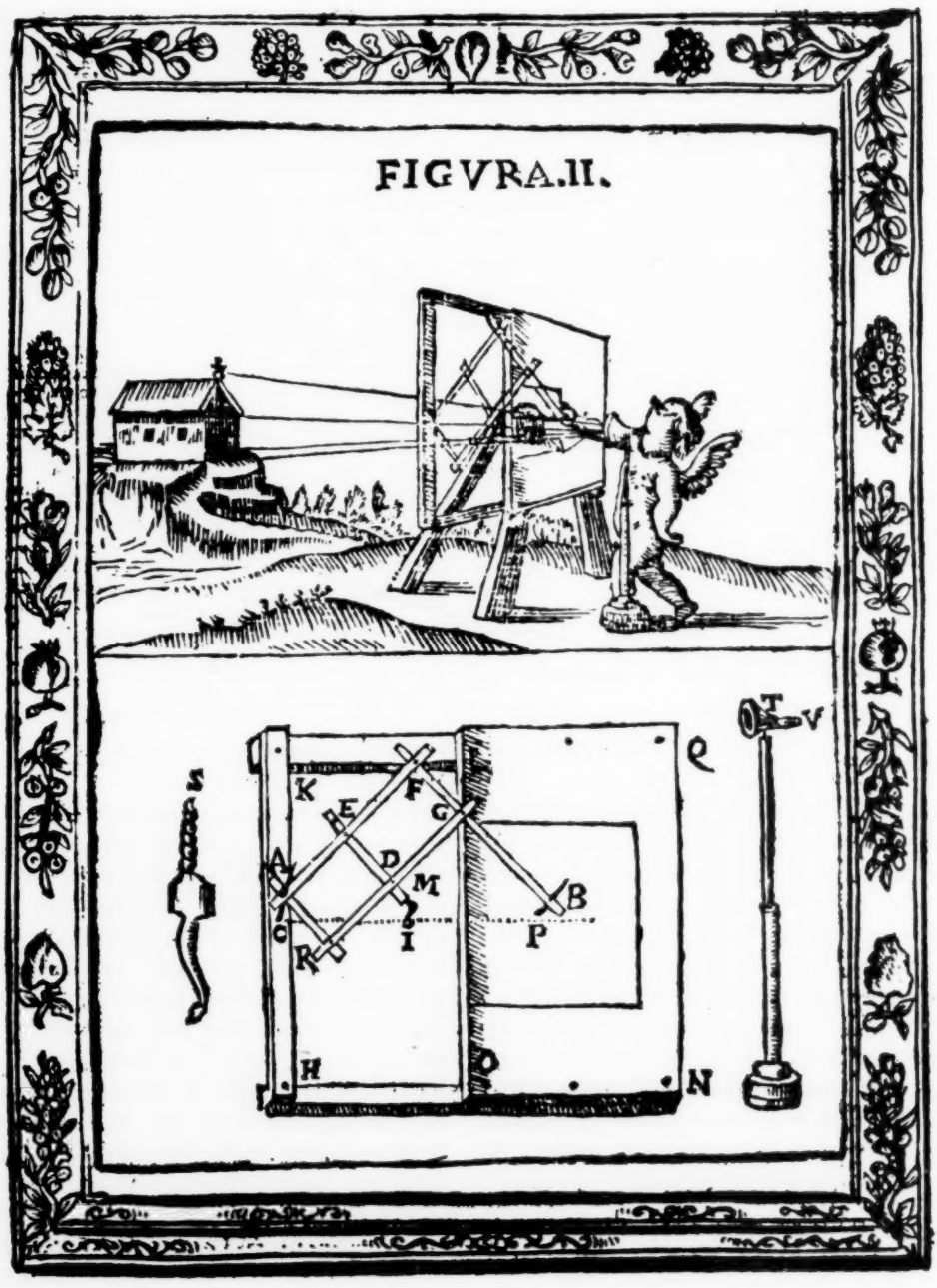


FIG. 9.

points of the object. If we immediately specify the surface to be a vertical plane, the eye to be at a given height above a horizontal plane called a ground plane, and the object to be on the opposite side of the "picture plane" from the eye, the problem is then, in nearly its simplest form, to find the "picture" of the object, i.e., the intersection of the picture plane with the cone of rays. An early formulation of this problem and its solution by use of a pantagraph is shown in slide 2 (Fig.

9). This is from the book *Paradossi per Pratticare la Prospettiva* by Guilio Troili published in Bologna in 1683 (the first edition was dated 1672). This is interesting as one of the first uses of the pantagraph in perspective, although the pantagraph itself was first described by its inventor, Christoph Scheiner, in 1631, and Albrecht Durer, painter and geometer, had earlier described a different instrument for perspective drawing.

Let's further simplify our problem by seeking only the representation of a square which lies on the ground plane with its "front" side coincident with the picture. To solve this problem we need only these elementary mathematical-perspective ideas: (1) every *line* of an object determines with the eye a *plane* whose intersection with the picture is the desired perspective representation; (2) all these planes pass through the eye; (3) any set of parallel lines determines an axial bundle of planes whose axis will be a line through the eye parallel to the set of lines; (4) the intersection of this line with the picture determines a point to which the representations of the parallels will converge; (5) any point of the object which lies on the picture plane will be its own perspective representation; (6) lines in the ground plane perpendicular to the picture will then converge to the foot of the perpendicular from the eye to the picture (let's call this the center of the picture); (7) lines in the ground plane making a 45 degree angle with the picture will converge to one of the two points located on a horizontal line through the center point and at a distance on either side of it equal to the distance of the eye from the picture. (This follows from visualizing the isosceles right triangles formed with the perpendicular from the eye to the picture by the lines through the eye parallel to the 45 degree lines in the ground plane.) Let's call these points the distance points D_1 and D_2 (Fig. 10). The perspective $ABC'D'$ of the square $ABCD$ is then determined by joining both A and B to C, and then to D_1 and D_2 respectively. These latter two lines are representations of the diagonals of the square and by their intersections with the sides determine the vertices C' and D'. Time prevents discussion of the extension of this idea to the representation of other squares and then of cubes. However, the connection with coordinates in two and three dimensions is of pedagogical significance and likewise demonstrates one method of finding by exact mathematical construction the perspective of any object. The mathematical theory is fundamental in all photogram-

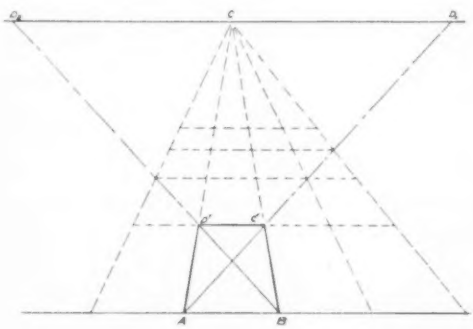


FIG. 10.

metric mapping[16] and has been put to work in designing letters for traffic signs which, drawn on the pavement, appear to stand up from the street as shown in this slide.[17] This latter use of perspective is a modern version of an old mathematical-perspective problem which with its solution is exemplified by Figure 11. In Figure 11 the trapezoid *abcd* when viewed with one eye located above *f* at a height *fg* is supposed to appear the same as square

[16] See for example: American Society of Photogrammetry, *Manual of Photogrammetry*, Pittman Publishing Co., 1944, Chap. VI, Geometric Characteristics, which is largely analytical in treatment, or Hart, C. A., *Air Photography Applied to Surveying*, Longmans Green and Co., 1940, Chap. VI, Elementary Perspective. An older and less analytical text is Deville, E., *Photographic Surveying*, Ottawa, Government Printing Bureau, 1895—see Chap. II, Perspective, and Chap. III, Perspective Instruments.

[17] From *The Detroit News*, July 23, 1939.

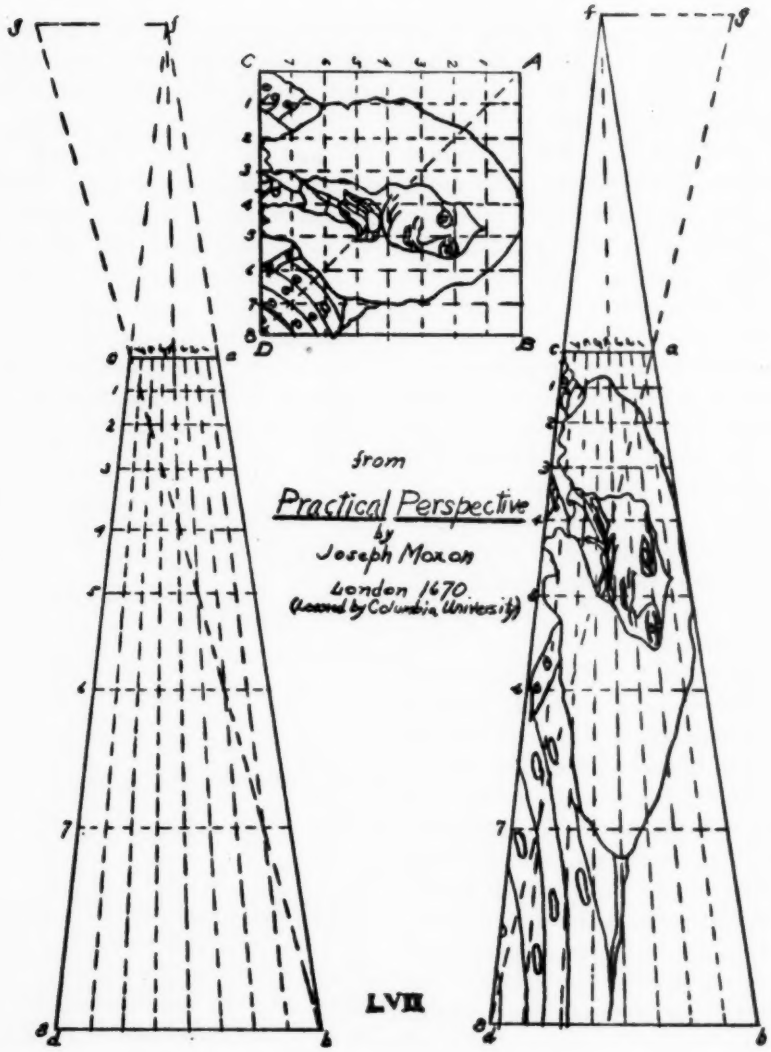


FIG. 11.

ABCD does if it is vertical and viewed from a point at a proper distance and height. This type of problem which, termed a "deformation," was popular at the time of Moxon's *Practical Perspective*, London, 1670, from which it was taken, is still today an interesting puzzle or recreational type of mathematics which has mathematical and scientific connections with such problems as projection, transformation of coordinates (or curves), maps, conformal mapping, airfoil theory, mathematical biology, and advertising, as shown in these slides.[18] The mathematical

[18] The first slide shown was from Sturm's *Mathesis Enucleata*, 1689, and the second was identical with Fig. 9, both from the University of Michigan Collection of Books on the History of Mathematics and Science. The fourth slide was identical with Fig. 11, and the fifth was similar to it and from the same source. The sources of the remaining slides were as follows:

3rd. Walker, C. L., *War Maps: Techniques and Secrecy*, Harpers, Sept., 1944, p. 375.

6th The Detroit News, July 23, 1939.

7th Advertisement by "167 Electric Light and Power Companies," *Life* Magazine.

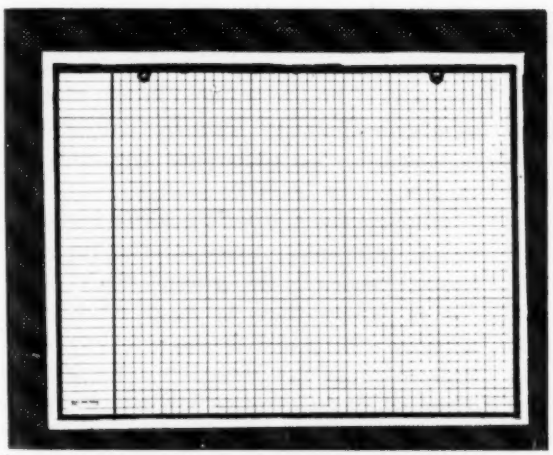
8th Deitz, C. H. and Adams, O. S., *Elements of Map Projection*, Special Publication No. 68. U. S. Department of Commerce, Coast and Geodetic Survey.

theory of perspective has even been used by some artists, witness the preliminary sketch made by Thomas Eakins for one of his paintings as shown by this picture from *Life* magazine for May 15, 1944, on page 77.

9th Thompson, D'Arcy, *On Growth and Form*, Macmillan, 1943, p. 1064.

10th Glauert, H., *The Elements of Aerofoil and Airscrew Theory*, Macmillan, 1944, p. 62.

How and where and for what grade levels these devices are to be used, whether in the mathematics club, as enrichment material for classes, as projects for students, or as daily teaching aids, I leave to you all as individuals. I believe they embody concreteness, reality, and substantial mathematics, all of which are helpful in teaching.



Something New!

"PLASTIC-PLATED" GRAPH-CHARTS

50" by 38"

with Metal Eyelets for Hanging

You Write With Wax Crayons and Erase with Dry Cloth

Ruled in one-inch squares with every fifth line heavier. Space down left hand side for names, abscissa, legends, etc. Used in arithmetic, algebra, physics, economics, commercial geography, industrial arts, agriculture, domestic science, for contests, Boards of Education meetings, attendance records, etc.

Printed in black on a clear white background. The transparent plastic surface provides a writing surface of glass-like smoothness on which one may write or draw with wax crayons.

The surface can be wiped clean and white with a soft dry cloth as frequently as desired. Washable inks can be used and removed with soap and warm water.

They are mounted on 3/16" laminated mounting boards and are durably bound with extra heavy tape. The outlines are beneath the plastic surface so they do not wear off.

No. 21 Graph—Plain. Has graph on one side and plain white "plastic plated" surface on other side. (You can make your own charts on this with India ink.)

No. 22 Graph—Graph. Has graph charts on both sides.

PRICES—$7.75 Each plus Transportation

Ideal crayons—Box of 5 colors (best to use) 35¢

Write for our illustrated Catalogue of other "Plastic-plated" maps and charts.

P. O. Box 2606. **Modern School Products** Cleveland, Ohio.

Please mention the MATHEMATICS TEACHER when answering advertisements

◆ THE ART OF TEACHING ◆

Provision for Individual Differences in High School Mathematics Courses

By WILLIAM LEE

Santa Monica High School, Santa Monica, California

How to maintain satisfactory standards of achievement in high school mathematics courses, as a larger and larger proportion of the population enrolls in high school with consequent lowering of the median ability, has long been recognized as a perplexing problem.

There is a tendency at present to advocate the requirement of one or two years of mathematics for all high school pupils, to be followed by elective courses for those who, as it is often put, can profit by further work in mathematics. (See W. D. Reeve, "Mathematics in the Post-War Period," *Scripta Mathematica*, vol. XI, 1945 pp. 275–307.)

At SBCHS, our attitude is that *every* pupil can profit by *four* years of mathematics, *provided* the courses offered are geared to his needs and abilities. Four years of mathematics of a caliber to challenge the superior pupil obviously must be provided in high school if leadership in mathematics is to be developed. But the pupil of low ability can hardly be consigned to limbo because he shows next to no mastery of even the most simple arithmetic; rather, it is he who should be urged to continue throughout high school in courses designed to develop in him at least the minimum of basic mathematical skills and understandings essential to good citizenship. Otherwise we, as educators, must make one of two embarrassing admissions: either (1) we are not sincere in our claim that a certain level of functional competence in mathematics is essential to good citizenship in the modern world; or (2) we are not willing to take the trouble to teach the group which most needs teaching, i.e., the pupils who find even "simple" mathematics difficult to learn.

The proportion of mathematical morons in the population is probably no greater than the proportion of any other kind of morons; and most of these, even with present indiscriminate admissions, do not enter high school. The high school pupil who rates low in mathematical achievement usually does so because of inadequate previous instruction. Let us not cull him out as hopeless; rather, let us provide him with instruction designed to overcome the neglect he has suffered in the past, in whatever quantity may be requisite to accomplish that objective. The requisite quantity may be large.

Homogeneous grouping seems essential in any attempt to tailor instruction to fit pupils' needs. In the present era of large classes, few teachers are so gifted as to be able to carry on instruction at two or three different levels within the same group. Even when the attempt is made, it would seem self-evident that time devoted to one subgroup must be denied another, so that the process inevitably becomes one of mere supervised self-instruction; and no doubt in many instances the words "supervised" and "instruction" would both be gross exaggerations.

In the May 1946 issue of THE MATHEMATICS TEACHER (p. 206), G. E. Hawkins outlines a plan for sectioning pupils into Minimum, Regular, and Special sections. But such a plan would seem to be of little value if Minimum, Regular, and Special Algebra, for instance, are all called Algebra, include fundamentally the same units of work (differentiated only in degree of

dilution) and are graded on the same grading scale. Why laboriously segregate pupils of D ability into a minimum section, when they could just as well earn their D's in a heterogeneous class without being deprived of the possible stimulus of exposure to the work of gifted pupils? If segregation is to be done at all, its justification (apart from reducing the wear and tear on the teachers' nerves) would appear to be to put each pupil in a situation wherein he would have a possibility of success.

If classes of segregated abilities are scheduled indiscriminately throughout the day, the mathematics department must compromise with other departments on something less than an ideal segregation program. At SBCHS (total enrollment about 500) we use a plan of homogeneous grouping which satisfies the needs of the mathematics department, without interfering with the grouping programs of other departments in any way. The plan is as follows:

FRESHMEN

In our seven-period day, three periods are set aside for Freshman mathematics. All ninth-graders are required to register in one of these three periods, while no Upperclassmen are permitted in any of these Freshman periods. Once registered in a given period, the pupils are segregated into two classes: Algebra, and General Mathematics (called Social Mathematics).

Since our entering Freshmen are all rather low in arithmetic ability (median ability of about sixth grade according to the tests we use) the initial segregation is made on the basis of IQ and pupils' preferences. No pupil is denied the opportunity to have a try at algebra; all those with an IQ of 100 or above are urged to remain in the Algebra section.

As the year progresses the stragglers in the Algebra sections, those who obviously cannot or will not maintain required standards of work in algebra, are transferred to the Social Mathematics sections, with the advice that they try algebra again a later year if they really want credit in algebra for college entrance or for some other reason. Since this involves no change of period, it does not interfere with the pupils' other classes in any way.

At the end of the course, when final grades are made out, pupils who have not done passing work in algebra can legitimately be given passing grades in Social Mathematics, provided their attendance has been acceptable and they seem to have put forth reasonable effort throughout the course—since development of functional competence in arithmetic is a prime objective in both classes. Thus there is practically no such thing as a "flunk" in Algebra at SBCHS; yet no algebra credit is granted unless satisfactory mastery of the subject has been achieved.

Our Algebra course includes the traditional topics, through factoring and an introduction to the quadratic formula; and with an unremitting attempt throughout to encourage the pupil to recognize that there is no sharp line of separation between algebra and arithmetic. The fundamental arithmetic operations are constantly held in the foreground, and analyzed in terms of algebraic processes.

The Social Mathematics course stresses understanding of arithmetic: "carrying" in addition, "regrouping" (*not* "borrowing") in subtraction, "indenting" in multiplication are analyzed and understood rather than remaining mere rote operations to be performed blindly. Simple equations are studied and used whenever convenient, particularly as a reasonable method for use in solving percentage problems, thus doing away entirely with the confusing and bewildering "Three Cases" of per cent. Intuitive geometry is stressed (this is also done in the Algebra sections). The problem material throughout is drawn from "socially useful" topics, hence the name of the course.

In the year 1945–46, our Freshman Algebra and Social Mathematics classes made about equal gains in general mathematical power as shown by the Schorling-Clark-Potter "Hundred-Problem Arithmetic Test" and the Foust-Schorling "Test of Functional Thinking in Mathematics."

Senior High School

While only one year of mathematics is absolutely required in our four years of high school, all pupils are very strongly urged to undertake at least two years and preferably four, so that the voluntary registration of Sophomores in mathematics is practically 100%. Two groups of second-year mathematics courses are provided:

(1) Algebra and Basic Mathematics. Two periods in the day are devoted to this program, for pupils who received credit in Social Mathematics during their Freshman year or who transfer from other schools where they studied no mathematics at all. Segregation is accomplished according to the plan already described, except that pupils' previous mathematics grades, together with ability as shown by the Hundred-Problem and Functional Thinking tests, are used instead of IQ. The Algebra is the same as that offered to Freshmen. The Basic Mathematics course includes more of simple algebra than does the Social Mathematics, and the problem material is somewhat more technical, being drawn largely from Shop, Commercial, and Aviation Mathematics. Short units on numerical trigonometry, logarithms, and the slide rule are introduced.

(2) Demonstrative Geometry and Intuitive Geometry. The intuitive geometry course is listed as "Measurement and Form" on the pupil's record, to avoid any possibility of its being interpreted as Geometry credit for college entrance purposes. We find only one period to be necessary for this grouping, consisting of two Demonstrative Geometry classes and one class of Measurement and Form. Credit in Algebra is prerequisite for admission to Demonstrative Geometry; credit in either Algebra or Basic Mathematics, or a similar course in another school, is prerequisite to Measurement and Form. Here again the desirability of having these courses grouped in one period is evident. Pupils who have received credit in Algebra are urged to attempt Geometry; those who are unable to maintain high standards can be transferred to Measurement and Form without upsetting their other high school registration. Furthermore, as in Algebra, it is practically never necessary to fail a pupil in Demonstrative Geometry; instead, credit can be granted in Measurement and Form.

The Demonstrative Geometry course stresses logical thought; the deductive method is applied to algebraic as well as geometric proofs. Three-dimensional geometry is included, not as a distinct subject but as a natural extension of the geometry of two dimensions, and non-Euclidean geometries are introduced. Sine, cosine and tangent relations are reviewed in connection with use of the transit, sextant, and other available instruments. Little time is spent on computation, as the pupils in this course already rate high in arithmetic, and it is only necessary to maintain and apply skills previously developed. These pupils however show excellent gains in the Hundred Problem and Functional Thinking tests.

The course in Measurement and Form consists of intuitive presentation of the important theorems of plane geometry, with emphasis on computation and a minimum introduction to the deductive method. A fairly long unit on map projections and aeronautics is included. A laboratory method is particularly fruitful in this course, pupils of this ability level appearing to learn best by doing.

(3) One period of the day is devoted to a class in Intermediate Algebra and Trigonometry, and a class in Solid Geometry and Advanced Algebra, for the high-ability pupils who desire these courses. This arrangement completely eliminates the pressure to enroll pupils in trigonometry before they have completed a third semester of algebra, which is inevitably exerted in some cases by registration officials when trigonometry and intermediate algebra are offered at different periods in the day.

Commercial Arithmetic and Mechanical Drawing are offered independently of the mathematics department.

This plan seems to eliminate entirely the social stigma which is sometimes attached to pupils in "minimum" sections. Rather, our problem is to persuade some of the pupils who have high ability, but who do not plan to attend college, that the intensive courses in algebra and geometry are better fitted to their needs than are the non-college-preparatory courses.

This mathematics offering provides the following four-year program for a pupil of low ability in mathematics:

Social Mathematics
Basic Mathematics
Measurement and Form or Mechanical Drawing
Algebra or Commercial Arithmetic

The pupil of good ability may elect the traditional sequence:

Algebra
Plane Geometry
Intermediate Algebra and Trigonometry
Solid Geometry and Advanced Algebra

There are many possible combinations between these two extremes. That the program meets with pupils' approval is shown by the fact that 75% of all Upperclassmen have enrolled voluntarily for mathematics (exclusive of Mechanical Drawing) for the year 1946–7, in addition to the required 100% enrollment of the Freshman class.

What Is the National Council of Teachers of Mathematics?

The National Council of Teachers of Mathematics is a country-wide association, organized on February 24, 1920, at Cleveland, Ohio, incorporated on April 28, 1928, under the laws of the State of Illinois, to assist in promoting the interest of mathematics in America, especially in the elementary and secondary fields, by holding meetings for the presentation and discussion of papers, by the publication of an official journal, monographs and yearbooks, of which the current one is the nineteenth. In this way the Council hopes to vitalize and coordinate the work of the many local organizations of teachers of mathematics, for the attention and consideration of the educational world.

The National Council of Teachers of Mathematics now has approximately 6000 members. Membership in the Council entitles one to all the privileges of membership, and one year's subscription to the official journal, THE MATHEMATICS TEACHER, which is published monthly except in June, July, August and September. There is at least one state representative in every State of the Union, whose duty it is to get new members and aid in renewals.

Any person who is interested in the field of mathematics is eligible for membership in the Council by paying the annual dues of $3. There are no formal requirements.

Any local club of mathematics teachers having at least fifteen members belonging to the National Council may become affiliated with the National Council upon passing a resolution to that effect and filling out the proper blank of affiliation supplied by the Secretary. We now have many affiliated organizations from New York to California, and from Louisiana to Minnesota. An engraved certificate is issued to each club without cost. There is no fee for affiliation.

The following benefits accrue from membership in the Council:

1. Any teacher who is a member of the National Council has thereby increased his professional standing.
2. By reading THE MATHEMATICS TEACHER he becomes acquainted with the problems of teaching mathematics in our public schools.
3. Through the official journal he becomes acquainted with new books that are ever being published in the field of mathematics.
4. In attending our annual meetings he has an opportunity to become personally acquainted with the leaders in the field.
5. He has an opportunity for self-expression either at our annual meetings or through the pages of our official journal.

IN OTHER PERIODICALS

By NATHAN LAZAR

Midwood High School, Brooklyn 10, New York

The American Mathematical Monthly

April 1947, Vol. 54, No. 4

1. Sewell, Colonel W. E., "Mathematics in the Army Education Program," pp. 195–200.
2. Polya, G., "Estimating Electrostatic Capacity," pp. 210–206.
3. Agnew R. P. and Walker, R. J., "A Trigonometric Infinite Product," pp. 206–211.
4. Goormaghtigh, R., "Orthopolar and Isopolar Lines in the Cyclic Quadrilateral," pp. 211–214.
5. Mathematical Notes, pp. 215–219.
   a. Sandham, H. F., "An Approximate Construction for e."
   b. Thomas, J. M., "The Trajectory in Vacuo."
   c. Hummel, P. M., "Note on Interpolation."
6. Classroom Notes, pp. 219–222.
   a. Hawthorne, F., "Derivation of the Equations of Conics."
   b. Wagner, R. W. "A Substitution for Solving Trigonometric Equations."
   c. Hoyt, J. P., "Evaluation of a Trigonometric Integral."
7. Elementary Problems and Solutions, pp. 223–231.
8. Advanced Problems and Solutions, pp. 232–238.
9. Recent Publications, pp. 238–242.
10. Club and Allied Activities, pp. 242–243.
    a. Increasing Student Interest in Mathematics.
11. News and Notices, pp. 244–248.
    a. The Institute of Mathematical Statistics.
    b. Summer Courses.
    c. Examination Announced for Statistician Positions.
    d. Personal Items.

The Duodecimal Bulletin

June 1947, Vol. 3, No. 2

1. Pitman, Isaac, "Reckoning Reform," pp. 1–5.
2. Beard, Ralph H., "Disingenuous Dissuasions," pp. 6–9.
3. Robert, Jr., Harry, C., "To Calculate the $\text{Log}_{12}2$," pp. 10–12.
4. Terry, George S., "Finding Square Roots," pp. 12–13.
5. Terry, George S., and Robert, Jr., Harry C., "Square Sums of Consecutive Squares," pp. 14–15.
6. Robert, Jr., Harry C., "Prime and Factor Tables," pp. 16–19.
7. "Notes on the Arabic Numerals," p. 20.
8. Beard, Ralph H., "Periods of Primes," pp. 21–22.
9. Terry, George S. "Periods and Submultiples of Reciprocals of Primes," p. 23.
10. "The DeFrance Algorithm for Finding the Length of Period of Reciprocals of Primes," pp. 23–25.
11. Lloyd, Mary, "Mathematical Recreations," pp. 25–26.
12. The Mail Bag, pp. 26–28.

The Mathematical Gazette

February 1947, Vol. 31, No. 293

1. "The First English Euclid," p. 1.
2. "Geometrical Drawing in The Higher School Certificate Course in Schools," pp. 2–20.
3. Finney, D. J., "Statistical Science and Agricultural Research," pp. 21–30.
4. Smith, C. A. B., "The Counterfeit Coin Problem," pp. 31–39.
5. Williamson, R. S., "Dot, the Fairy: An Introduction to Decimals," pp. 40–41.
6. Mathematical Notes (1942–1948), pp. 42–50.
7. Reviews, pp. 51–64.
8. Gleanings far and near (1511–1528) pp. 20, 30, 39, 41, 64.

School Science and Mathematics

April 1947, Vol. 47, No. 4

1. Sleight, Norma, "More Fun with Conics," pp. 303–304.
2. Schaaf, William L., "Some Sidelights for the Arithmetic Teacher," pp. 315–322.
3. Bell, E. T., "The Demise of Euclid," pp. 327–335.
4. Brickley, Raymond S., "Teaching Algebra," pp. 341–347.
5. Horr, Raymond, "The Signs of Trigonometric Functions Again," p. 382.
6. Problem Department, pp. 383–388.
7. Books and Pamphlets Received, pp. 389–390.
8. Book Reviews, pp. 390–396.

Scripta Mathematica

September 1946, Vol. 12, No. 3

1. Richards, John F. C. B., "Boissiere's Pythagorean Game" (Rithmomachia), pp. 177–217.
2. Book Reviews, pp. 219–222.
3. Recreational Mathematics.
   a. Grossman, Howard D., "Fun with Lattice Points," pp. 223–225.
   b. Gloden, A., "Magic Squares," pp. 225–226.
4. Miscellaneous Notes.
   a. Karapetoff, Vladimir, "Graphical Determination of the Most Probable Plane," pp. 227–230.
   b. Goldberg, Oscar, "On Numbers in the Bible," pp. 231–232.
5. Curiosa.
   128. Van der Pol, B., "Divisors and Partitions," p. 218.
   129. Thebault, Victor, "Number Pleasantries," p. 218.
   130. Whitlock, Jr., W. P., "A Problem in Representations," p. 218.
   131. Moessner, Alfred "Simultaneous Diophantine Equations," p. 226.
6. Quotations.
   15. Bergmann, S. H., "A Kantian Interpretation of Science."

NEWS NOTES

The Annual Spring Conference of Mathematics Teachers of Minnesota was held at The University of Minnesota, Coffman Memorial Union on April 25 and 26, 1947

PROGRAM

FRIDAY, APRIL 25, 1947

2:00 P.M. Films
"Story of Money"
"The Circle"
"Similar Triangles"
3:00 P.M. Film Strips
"Geometry in Art"
"Deductive Reasoning"
"Optical Illusions"
"Timekeeper Through the Ages"
3:00 P.M. Coffee Hour
Host: Minneapolis Mathematics Club
4:00 P.M. The Enrichment of Mathematics Instruction
Speaker: Mr. Paul Trump, University of Wisconsin
Discussion and announcements.

SATURDAY, APRIL 26, 1947

9:30–10:45 A.M. Films and Film Strips (Same as the Friday showing)
10:45 A.M. "What are the Current Trends in Improving the Mathematics Curriculum?"
Chairman: Mr. Walter Andrews, State Department of Education
Participants and Topics:
"Mathematics in the Core Curriculum," Mr. Clifford Bertram, Principal, Floodwood
"General Mathematics," Miss Edna Norskog, Alexandria
"Sequential Mathematics," Dr. George C. Priester, University of Minnesota
"Mathematics for General Education," Mr. Robert S. Gilchrist, Minneapolis Curriculum Specialist
Comments by: Mr. Lee Brueckner, Mr. Paul Trump, and Mr. Donovan Johnson
12:30 P.M. Luncheon
Toastmistress: Miss Dela Hunt
Speaker: Mr. Paul Trump, "Fact or Fancy."

The Second Annual Conference for Southwest Louisiana High School Mathematics Teachers was held at Southwestern Louisiana Institute, Lafayette, La. on May 2, 1947.

FIRST SESSION

Z. L. Loflin, Head of the Department of Mathematics, S. L. I. presiding
9:30 A.M. Teaching High School Students to Solve Geometry Originals—Miss Lurnice Begnaud—Lafayette High School.
9:45 A.M. A Unified Effort in Teaching Mathematics—Z. T. Gallion, Jr., Department of Mathematics, S. L. I.
10:10 A.M. Teaching Geometry for Logic—Miss Jessie Mae Hoag, Department of Mathematics, S. L. I.
10:30 A.M. Coffee—F. H. Hamilton Training School Cafeteria.
11:00 A.M. Audo-Visual Aids in Teaching Mathematics.
12:30 P.M. Luncheon—Toby's Oak Grove.

SECOND SESSION

Z. T. Gallion, Jr., Department of Mathematics, S. L. I. presiding
2:00 P.M. Enriched Teaching of High School Algebra—Dr. Margaret Rae Davis—Southeastern Louisiana College.
2:30 P.M. Discussion.
3:15 P.M. Appraisal—Z. L. Loflin, S. L. I.
3:30 P.M. Adjournment.

The 24th Annual Joint Meeting of the Louisiana-Mississippi Section of the Mathematical Association of America and the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was held at Mississippi Southern College in Hattiesburg on Friday and Saturday, April 25 and 26, 1947.

THE LOUISIANA-MISSISSIPPI SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

Room 111, College Hall—2:30 P.M.,
Friday, April 25
W. V. Parker, Louisiana State University, Chairman, Presiding
F. C. Gentry, Louisiana Polytechnic Institute, Secretary
Announcements and Appointment of Committees

1. Installment Buying—Irby C. Nichols, Louisiana State University.
2. Mathematics Applied to Meterology—J. T. Lee, Mississippi Southern College.
3. Problems Concerning Volumes—C. D. Smith Mississippi State College.
4. Three Cubic Loci—F. C. Gentry, Louisiana Polytechnic Institute.
5. Discussion of Report of Association Committee for the Coordination of Studies on Mathematical Education.
   Part 1.—L. M. Garrison, Louisiana Polytechnic Institute.
   Part 2.—T. A. Bickerstaff, University of Mississippi.
   Part 3.—Z. L. Loflin, Southwestern Louisiana Institute.

Reception

Given by Mississippi Southern College—4:15 P.M.
President's Home, M. S. C. Campus

Joint Banquet (Informal)

Venetian Room, Forrest Hotel—7:00 P.M.,
Friday, April 25
PROGRAM
W. L. Johnson—Toastmaster
Music
Welcome—Dean R. A. McLemore, Mississippi Southern College Supt. Steve Blair, Hattiesburg Public Schools.
Response—Miss Jessie Mary Hoag, Southwestern Louisiana Institute—For the National Council.

B. E. Mitchell, Millsaps College—For the Association.
Address: Mathematics Teaching—Tulane Brand —H. E. Buchanan, Tulane University.

LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
Library, M.S.C. Demonstration School—8:30 A.M.
Saturday, April 26th, 1947
Miss Virginia Felder, Moorhead, Miss., Chairman, Presiding
Mrs. H. P. Smith, Monroe, Louisiana, Secretary

1. Report of the Atlantic City Meeting—Houston T. Karnes, Louisiana State University.
2. A Unified Effort in Teaching Mathematics—Z. T. Gallion, Southwestern Louisiana Institute.
3. The Aims of a Junior College Mathematic Curriculum—Miss Lurline Stewart, Hinds Junior College.
4. Teaching High School Students to Solve Geometry Originals—Miss Lurnice Begnaud, Lafayette (Louisiana) High School.
5. Report of the Joint Committee—W. L. Duren, Tulane University.
6. Election of Officers.

THE LOUISIANA-MISSISSIPPI SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA
Room 111, College Hall, 10:30 A.M., Saturday, April 26
W. V. Parker, Louisiana State University, Chairman, Presiding
F. C. Gentry, Louisiana Polytechnic Institute, Secretary

Address: A Foundation for the Point-Calculus of Grassman—H. L. Smith, Louisiana State University.
Business Meeting and Election of Officers.

OFFICERS 1946–47

LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

J. H. Banks, East Central Junior College, Decatur, Miss.—Chairman.
Miss Virginia Felder, Moorhead, Mississippi—Vice-Chairman.
Mrs. H. P. Smith, Monroe, Louisiana—Secretary.
Houston T. Karnes, Louisiana State University—Recorder.
Miss Jessie Mary Hoag, Southwestern Louisiana Institute—Louisiana Representative.
Miss Lurline Stewart, Hinds Junior College, Raymond, Mississippi—Mississippi Representative.

LOUISIANA-MISSISSIPPI SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

W. V. Parker, Louisiana State University—Chairman.
W. L. Johnson, Mississippi Southern College—Vice-Chairman.
Z. L. Loflin, Southwestern Louisiana Institute—Vice-Chairman.
F. C. Gentry, Louisiana Polytechnic Institute—Secretary.

Newly elected officers of the Louisiana-Mississippi sections of the National Council of Teachers of Mathematics and the Mathematical Association of America are:

National Council of Teachers of Mathematics:
Miss Lurnice Begnaud, Lafayette, Louisiana High School, Chairman.
Mr. G. E. Miller, Northwestern State College, Natchitoches, Louisiana, Vice-Chairman.
Miss Mary Morphis, Hattiesburg High School, Secretary.
Dr. Houston T. Karnes, Louisiana State University, Recorder.
Miss Jessie May Hoag of Southwestern Louisiana Institute at Lafayette is the Louisiana representative of the Council and Miss Lurline Stewart of Hinds Junior College at Raymond is the Mississippi representative.

Mathematical Association:
Mr. W. L. Johnson, Mississippi Southern College, Chairman.
Mr. T. A. Bickerstaff, University of Mississippi, Vice-Chairman.
Mr. Z. L. Loflin, Southwestern Louisiana Institute, Vice-Chairman.
Mr. F. C. Gentry, Louisiana Polytechnic Institute, Ruston, Louisiana, Secretary.

The Mathematical Association of America's Metropolitan New York Section held its Sixth Annual Meeting at Pratt Institute, Brooklyn, N. Y. on Saturday, April 19, 1947.

PROGRAM

MORNING SESSION, 10:00 A.M.
Chairman: Professor W. H. H. Cowles, Pratt Institute

Address of Welcome
Dean Nelsen S. Hibshman, School of Engineering, Pratt Institute.
The Mathematics of Magic Squares
Mr. Harry Sitomer, New Utrecht High School
Elementary Geometry as an Algebraic System
Professor Walter Prenowitz, Brooklyn College.
Mathematics in Psychology
Dr. Lloyd H. Beck, Yale University

LUNCHEON IN THE PRATT INSTITUTE CAFETERIA 12:30 P.M.

AFTERNOON SESSION, 2:00 P.M.
Chairman: Mr. Morris Hertzig, J.H.S. 252 Brooklyn

What Statistics, if Any? in a Required General Mathematics Course? Report of an Attack on this Problem at Queens College—Professor T. F. Cope, Queens College.
A Report on High School Mathematical Preparation—Professor F. H. Miller, Cooper Union.
The Tentative Secondary School Syllabus in Mathematics for Grades 7 Through 12—Mr. Joseph Orleans, George Washington High School.

All present are invited to remain for tea at The Women's Club, 229 Clinton Avenue as the guests of The Faculty Wives' Club of Pratt Institute.

Officers of the Metropolitan New York Section, 1946–1947

Howard E. Wahlert, New York University—Chairman.
W. H. H. Cowles, Pratt Institute—Vice Chairman.
Morris Hertzig, J.H.S. 252 Brooklyn—Vice Chairman.
Carl B. Boyer, Brooklyn College—Secretary.
Aaron Shapiro, Midwood High School—Treasurer.

The High School Mathematics Conference of Louisiana Polytechnic Institute was held at Ruston, Louisiana on Friday, May 9, 1947.

PROGRAM

Little Theater, Howard Auditorium—9:30 A.M.

1. Welcome—Dean H. L. Hughes, School of Arts and Science, Louisiana Polytechnic Institute.
2. Some Needed Changes in Secondary Education—Dean G. W. Bond, School of Education, Louisiana Polytechnic Institute.
3. Guidance in High School, With Special Emphasis on Mathematics—C. T. Woodward, Supervisor, Jackson Parish.
4. Panel Discussion—Dr. P. K. Smith, Chairman.
   (1) What Constitutes a Good High School Mathematics Teacher?—H. F. Schroeder, Louisiana Polytechnic Institute.
   Response—Miss Frances Wilson, Quachita Parish H. S.
   (2) Improving High School Mathematics Instruction—D. W. McCleish, Supervisor Bienville Parish.
   Response—Mrs. H. E. Richardson, Jonesboro-Hodge High School.
   (3) Modern Trends and Outlook in High School Mathematics—J. P. Causey, Principal Oak Ridge High School.
   Response—M. A. Price, Principal Bernice High School.
   (4) General Discussion Period—Dr. P. K. Smith.

LUNCH HOUR, LITTLE DINING HALL—12:30 P.M
Music
After Dinner Speaker—President Cottingham, Louisiana Polytechnic Institute

Room 122, Bogard Hall—1:30 P.M.

1. What the School of Engineering Expects of The High School Graduate—Dean Roy Sessums, School of Engineering, Louisiana Polytechnic Institute.
2. Greetings from The National Council of Teachers of Mathematics—Dr. F. C. Gentry, Louisiana Polytechnic Institute.
3. Business Session—Lester M. Garrison.
4. Mathematics Films.

The Association of Mathematics Teachers of New Jersey held its 85th regular meeting in Kirkpatrick Chapel at Rutgers University in New Brunswick, N. J. on Saturday May 3, 1947.

PROGRAM

EDUCATION AT THE CROSSROADS

10:00 A.M. Address: Mathematics is Making Headway in New Jersey—Madeline D. Messner, Abraham Clark High School, Roselle.

10:20 A.M. Address: Social Implications in the Teaching of Arithmetic—Thomas J. Durell, Assistant Commissioner of Education, State Department, Trenton.

10:50 A.M. Panel Discussion: Problems in Integrating the Teaching of Secondary School and College Mathematics.

Moderator and Discussion Leader, David R. Davis, State Teachers' College, Montclair.

Panel Members:

William L. Sterner, Belvidere High School, Belvidere.
George W. Garthwaite, Plainfield High School, Plainfield.
Emory P. Starke, College of Arts and Science, Rutgers University, New Brunswick.
Robert M. Walter, New Jersey College for Women, New Brunswick.

11:50 A.M. Business Meeting of the Association
Membership Report and Treasurer's Report—Mary C. Rogers.
Report of the Nominating Committee—Fred L. Bedford.
Election and Presentation of Officers for 1946–1947.

1:00 P.M. Luncheon at University Commons.

ASSOCIATION OFFICERS

President: Madeline D. Messner, Abraham Clark High School, Roselle.
Vice-Presidents: Robert M. Walter, New Jersey College for Women, New Brunswick; Leonard Lumb, East Side High School, Newark; Marian Moore, Roosevelt School, Kearny.
Secretary-Treasurer: Mary C. Rogers, Roosevelt Junior High School, Westfield.
Corresponding Secretary: Elizabeth Cullen, Hamilton Junior High School, Elizabeth.
Recording Secretary: Hubert B. Risinger, Davey Junior High School, East Orange.

PROGRAM COMMITTEE

Robert M. Walter, *Chairman*

Virgil S. Mallory	Mary C. Rogers
Albert E. Meder, Jr.	Madeline D. Messner
J. Dwight Daugherty	Amanda Loughren

EDITORIAL BOARD

Fred L. Bedford, *Chairman*

Leonard Lumb	M. Agnes Kelly

The California Mathematics Council held its Spring Conference for 1947 at J. Francis Polytechnic High School, 400 West Washington, Los Angeles.

PROGRAM

MONDAY, MARCH 31, 1947

9:45–10:30 A.M. Business Meeting

10:30–12:00 A.M. General Session

Topic: "Essentials of Mathematics for a General Education."

Presiding: Mr. Dale Carpenter, Supervisor of Mathematics, Los Angeles City Schools.

Speakers: Dr. C. C. Trillingham, Superintendent, Los Angeles County Schools; Dr. Cornelius H. Siemens, Director Compton Junior College; Mr. Millage G. Mont-

gomery, Assistant Superintendent, Los Angeles City Schools.

12:00–1:30 P.M. Lunch and recreation

1:30–3:00 P.M. Discussion Groups

Topic: "Teaching Mathematics for Meaning and Understanding."

1. *Elementary Group*

Chairman: Dr. Roy deVerl Willey, San Jose State College.

Discussion Leaders: Mrs. Else Thomas, Santa Barbara Schools. (Others to be announced.)

2. *Junior High Group*

Chairman: Mr. Howard E. Floyd, Pasadena City Schools.

Discussion Leaders: Mr. Francis B. Martin, Lynwood Junior High School; Mrs. Ruth O. Lana, Nightingale Junior High School. (Others to be announced.)

3. *Senior High and Junior College Group*

Chairman: Mr. Ernst A. Sundberg, Polytechnic High School.

Discussion Leaders: Mr. Bruce A. Casey, Polytechnic High School, Riverside.

TUESDAY, APRIL 1, 1947

9:45–10:30 A.M. Business Meeting

10:30–12:00 M. General Session

Topic: "Evaluation—What Should Good Mathematics Textbooks Contain? What Should Mathematics Tests, Test?"

Presiding: Miss Harriette B. Burr, San Jose High School.

Speakers: Dr. Peter L. Spencer, Professor of Education, Claremont Graduate College; Dr. Clifford Bell, Associate Professor of Mathematics, University of California at Los Angeles; Mrs. Howardine Hoffman, Director, Division of Elementary Education, Los Angeles County Schools.

12:00–1:30 P.M. Lunch and recreation

1:30–2:30 P.M. General Session

Topic: "Visual Aids in Mathematics."

Presiding: Miss Alice M. Phillipson, South Gate Junior High School.

Speaker: Mrs. Emma B. Sturtevant, South Pasadena, San Marino Junior High School. Motion Pictures and Discussion.

2:30–3:00 P.M. Reports of Discussion Group Chairmen

3:00–4:00 P.M. Executive Committee Meeting Adeline B. Newcomb, *President.*

Eighteen experts in the field of mathematics have made their first contributions to the Encyclopaedia Britannica, writing for the 1947 revised printing of the reference work, it was announced by Walter Yust, editor-in-chief.

Eric Temple Bell, professor of mathematics at California Institute of Technology at Pasadena, author of "Men of Mathematics," and "Development of Mathematics," has written or revised 23 articles for the new printing. Included were these topics: "Niels Henrick Abel," "Algorism," "Wolfgang Bolyai," "George Boole," "Augustin Louis Cauchy," "Arthur Cayley," and "August Leopold Crelle."

Four articles were written or revised for the new printing by Carl Benjamin Boyer, assistant professor of mathematics, Brooklyn College, Brooklyn, New York, author of "The Concepts of the Calculus: A Critical and Historical Discussion of the Derivative and the Integral," on these topics: "Raymond Clare Archibald," "Alexei Nikolaievich Krylov," "Henri Leon Le Besque," and "Rudolph Lipschitz."

The other new contributors in this field and the topics on which they wrote are as follows: George Robert Stibitz, consultant in applied mathematics, University of Vermont, "Mathematical Instruments"; Antoni Zymund, professor of mathematics, Mount Holyoke College, South Hadley, Massachusetts, author of "Trigonometric Series," "Fourier Series"; Tibor Rado, professor of mathematics at Ohio State University, Columbus, Ohio, "Conformal Representation"; Hubert Stanley Wall, professor of mathematics, Illinois Institute of Technology, author of "Analytic Theory of Continued Fractions," "Continued Fractions," and "Infinite Products."

Miss Vera Sanford, professor of mathematics at State Teachers College at Oneonta, New York, "Ludovici Ferrari," "Finger Numerals," "Hippocrates of Chios," and "Mathematical Notations."

Rudolph Ernest Langer, professor of mathematics and chairman of the mathematics department of University of Wisconsin, "Differential Equations," "Mensuration," and "G. D. Birkhoff."

Vernon Guy Grove, professor of mathematics, Michigan State College of Agriculture and Applied Science, East Lansing, Michigan, "Differential Geometry (Metric)," and "Differential Geometry (Projective)."

A completely new article on "Integral Equations," was prepared by Theophil Henry Hildebrandt, professor of mathematics and chairman of the department of mathematics of University of Michigan, Ann Arbor. Samuel Stanley Wilks, professor of mathematical statistics, Princeton University, author of "Mathematical Statistics," revised "Interpolation" for the Encyclopaedia Britannica.

Garrett Birkhoff, associate professor at Harvard University, author of "Lattice Theory," contributed the articles "Lattice Theory," and 'Foundations of Mathematics," for the new printing. Ernest Preston Lane, professor and chairman of the department of mathematics at the University of Chicago, author of "Projective Differential Geometry of Curves and Surfaces," prepared "Angle," "Involute," and "Involution."

Derriek Henry Lehmer, associate professor of mathematics at the University of California, Berkeley, California, wrote on "Bernouilli Numbers," and "Magic Square."

A. Adrian Albert, professor of mathematics at the University of Chicago, author of "Modern Higher Algebra," and "Structure of Algebras," wrote on the topics "Fields," and "Linear Algebras"; Raymond Walter Barnard, associate professor of mathematics, the University of Chicago, prepared the articles "Commutative Laws," "Infinitesimal" and "Mathematical Induction"; Harold Thayer Davis, Henry S. Noyes professor of mathematics and chairman of the department of mathematics at Northwestern University, author of "The Theory of Linear Operators," and "Tables of the Higher Mathematical Functions," wrote on "Mathematical Tables."

Aristotle Demetrius Michal, professor of

mathematics at the California Institute of Technology at Pasadena, author of "Matrix and Tensor Calculus with Applications to Mechanics, Elasticity and Aeronautics," revised the articles on "Differential Forms," and "Tensor Analysis."

Articles by more than 4,000 distinguished scholars and other authorities from 62 countries and about 550 universities, colleges and other scientific, cultural and educational institutions appear in the 1947 Britannica. A revised printing of the reference work has been published each year since 1932 when the policy of continuous revision replaced the old "edition" system. Under the new policy, approximately 35,000 new or revised articles have been contributed by authorities in 30 major fields of human knowledge.

Iowa Association of Mathematics Teachers Annual Meeting, Des Moines, Iowa, November 7, 1947

Miss Dora Kearney, *Pres.*
First Baptist Church
8th & High

Miss Hesther Douthart
Vice-Pres.

12:00 Noon—Luncheon

"Important Proposals for the Improvement of Mathematics." . . . Dr. Raleigh Schorling, University of Michigan.

2:00 PM.—An open discussion of the new Iowa State program in Mathematics

Leaders of Discussion:

1 Dr. H. Vernon Price, chairman—Iowa City
2 Miss Crystal Kolpin—Webster City
3 Miss Kathryn Hinsenbrock—Charles City
4 Mr. Edwin Mogck—Storm Lake
5 Mr. C. Freaman Reid—Dubuque

Please send $1.25 for your luncheon reservation to:
Mrs. Paul Norris, 4527 Waveland Court, Des Moines 12, Iowa on or before Nov. 3, 1947.

15th Yearbook

of the

National Council of Teachers of Mathematics

The PLACE OF MATHEMATICS in Secondary Education

The final report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics

Recently voted one of the 60 most outstanding Educational Books of 1940

THIS is the report that teachers of mathematics and educators hae been waiting for—the most important work in its field since the Report of the National Committee on Mathematical Requirements under the auspices of The Mathematical Association on *The Reorganization of Mathematics in Secondary Education* in 1923.

CONTENTS

Introduction: The Role of Mathematics in Civilization

I. Looking at Modern Education and Its General Aims
II. General Objectives for Secondary Education
III. The Place of Mathematics in Education
IV. The Mathematical Curriculum
V. One Organization of the Materials of Instruction, Grades 7-12
VI. A Second Curriculum Plan
VII. The Problem of Retardation and Acceleration
VIII. Mathematics in Junior College
IX. Evaluation of Progress in Mathematics Instruction
X. The Education of Teachers

Appendix: Analysis of Mathematical Needs. The Transfer of Training. Terms, Symbols, Abbreviations, Equipment. Selected References.

Price, $1.75 postpaid

BUREAU OF PUBLICATIONS

Teachers College *New York* *Columbia University*

Please mention the MATHEMATICS TEACHER when answering advertisements